

SWITCHING DYNAMICS

by

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Acknowledgement in Brief

"Well, the first days are the hardest days;
Don't you worry anymore.
'Cause when life looks like easy street,
there is danger at your door."

from "Uncle John's Band"
Music by Jerry Garcia,
Lyric by Robert Hunter,
Copyright 1970, 1973,
Ice Nine Publishing Co.

This work is dedicated to:

My parents, for years and years of loving support;

The Dream Street Crew, for a place to sleep (out of the rain);

Carver Mead, for helping me get into Caltech, for his faith in my
scientific abilities;

To Chuck Seitz, whose psychoanalytical skills helped me through
some rough spots (I'm catching onto your game, Seitz!)
who gave me a good push in a direction which has proved
utterly fascinating and it seems, fruitful;

To Vivian (Pretty Mama!), for typing and patience;

To all my friends;

And lastly, to the Dabney House Jug Band (Ted Waverka, Wert,
E**2, Stuart, the drum beaters and cymbal crashers,
Margie Farrell, and Barbara Turpin for incessant
listening, and of course Bruno (whoever he was) whose
hours and hours of funky, funny quasi-periodic
anharmonic oscillations provided welcome and needed
fellowship in times of trial.

Amen!!!

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CHAPTER 1

A BASIC EXAMPLE

Consider, to begin, a simple bistable device of the type commonly used to store information in all present day digital computers: the cross coupled inverter, or flip-flop (Fig. 1). The basic operation of this device is well known: we have feedback loops of positive gain by means of which a voltage at either node A or B can drive itself in a reinforcing way. In this way the information, a logical value of 0 or 1, is stored for an indefinite period of time as the voltage at node A or node B.

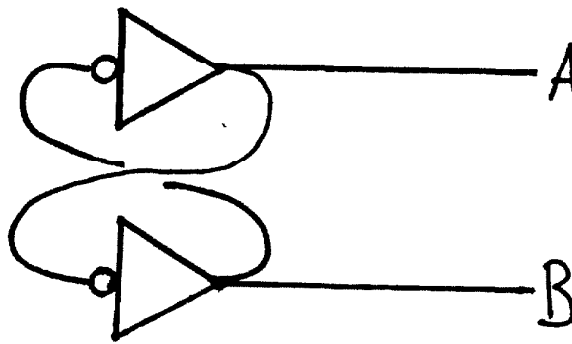


Figure 1. Cross-coupled inverters or flip-flop

If we want to make a simple mathematical model of such behavior we might proceed as follows: suppose the output voltages vary smoothly⁽¹⁾ between 0 and V_0 . Generally, we interpret the state of this system with $(V_A, V_B) = (V_0, 0)$ as a logical one stored at A and logical 0 at node B. The state $(V_A, V_B) = (0, V_0)$ has the obvious interpretation. If we plot the states of the system as a two-dimensional graph in the square⁽²⁾ $[0, V_0] \times [0, V_0]$ in the (V_A, V_B) plane, we can associate with each point (V_A, V_B) a vector (\dot{V}_A, \dot{V}_B) . This vector represents the time rate of change of the system when the system is in state (V_A, V_B) . That is, we suppose the dynamics of the flip-flop are given by a pair of functions F_A and F_B of (V_A, V_B) , such that

$$\dot{V}_A = F_A(V_A, V_B)$$

and

$$\dot{V}_B = F_B(V_A, V_B) .$$

Then what happens is: if the system is somehow forced into state (V_{A0}, V_{B0}) , it will evolve in time. Its trajectory in the phase plane will be a curve $(V_A(t), V_B(t))$ such that $V_A(0) = V_{A0}$ and $V_B(0) = V_{B0}$. At any time t ,

$$\frac{dV_A}{dt} = \dot{V}_A = F_A(V_A, V_B)$$

and

$$\frac{dV_B}{dt} = \dot{V}_B = F_B(V_A, V_B) .$$

Thus we can think of the vector field $F = (F_A, F_B)$ as the "driving force" of the flip-flop. Now no matter what detailed behavior one imagines F may have, there are certainly some properties F must have, if we are to use it in any model of a bistable device. Note we postulated $(V_0, 0)$ and $(0, V_0)$ as the stable, "storage modes," of our device. Presumably this means that if the device is initialized at one of these points, it will remain there indefinitely, i.e., its trajectory will be

$$V_A(t) = V_{A0} = \text{constant} ,$$

$$V_B(t) = V_{B0} = \text{constant} .$$

Then

$$\dot{V}_A = \dot{V}_B = 0, \text{ or}$$

$$F = 0$$

at the points $(0, V_0)$ and $(V_0, 0)$.

Thus the stable states of our bistable device must be zeroes of the vector field F .

Let us recapitulate what we have done so far: we have a fairly general dynamic model of a flip-flop, or bistable device, which has the following properties: (a) all voltages are in the range $[0, V_0]$, (b) any voltage pair at the nodes A,B is acceptable as an initial value provided each is the range $[0, V_0]$. There are various ways we can accomplish this in practice. The simplest is probably to put additional switches in the

circuit of Fig. 1 to break the feedback loop while the nodes A and B are being charged to initial values (V_{A0}, V_{B0}). Then we simultaneously switch the charging voltage off and the feedback loops on. The resulting circuits are shown in Figure 2.

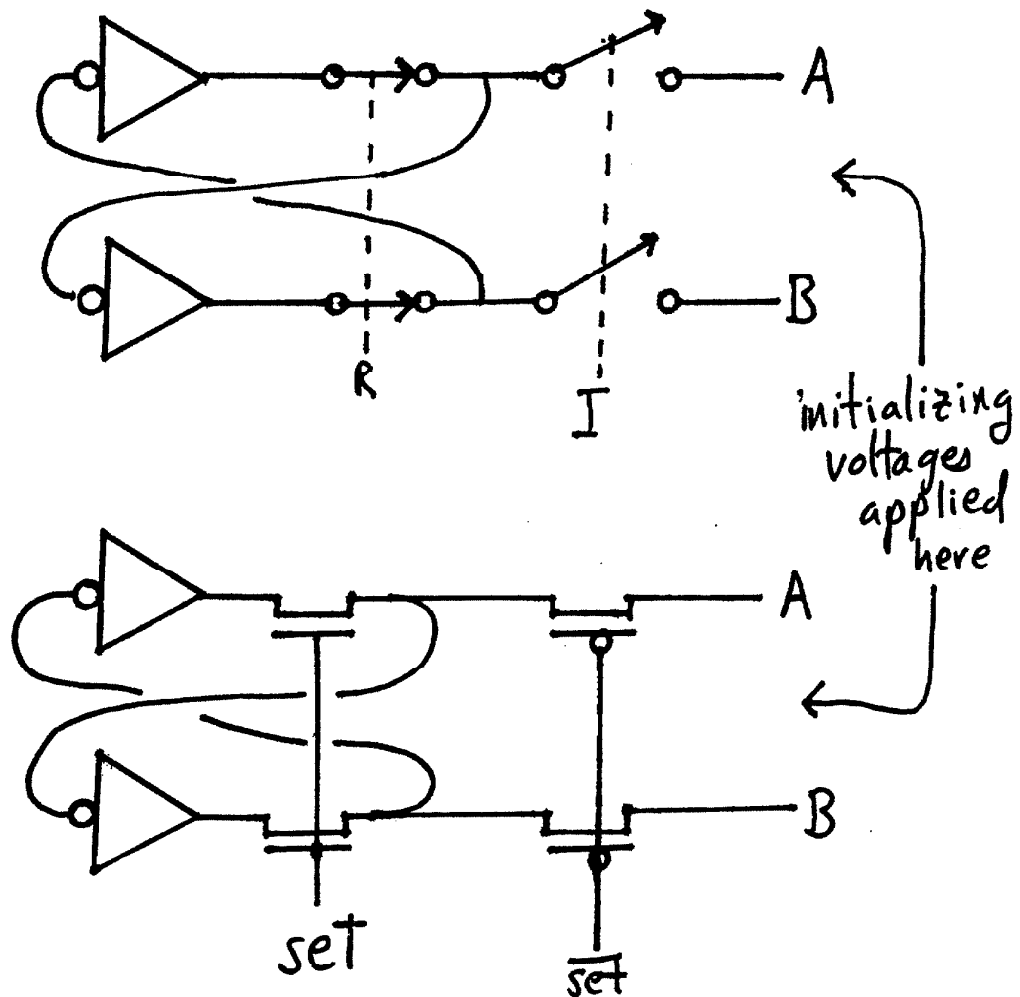
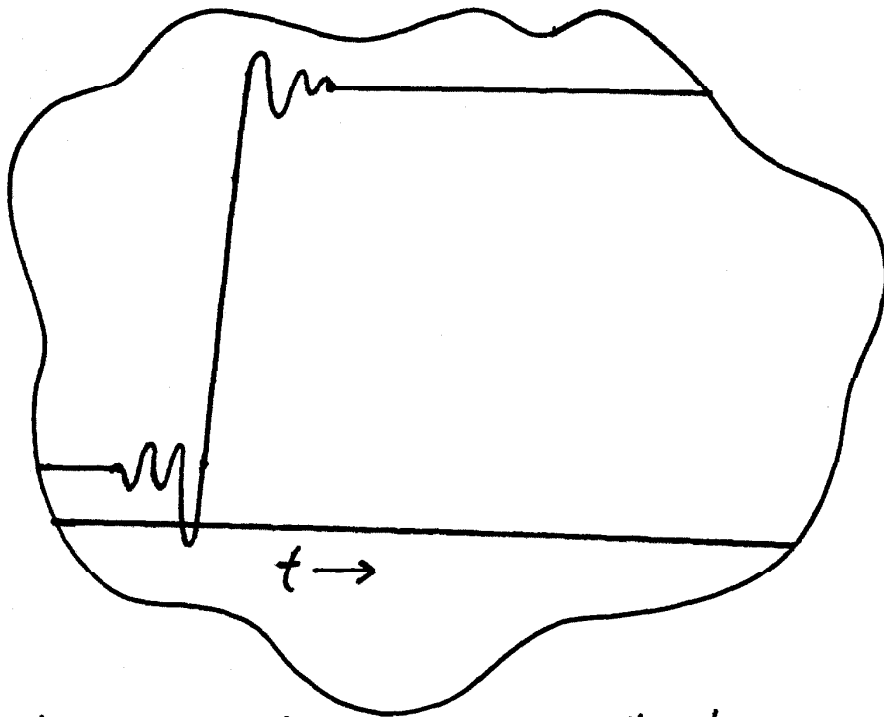
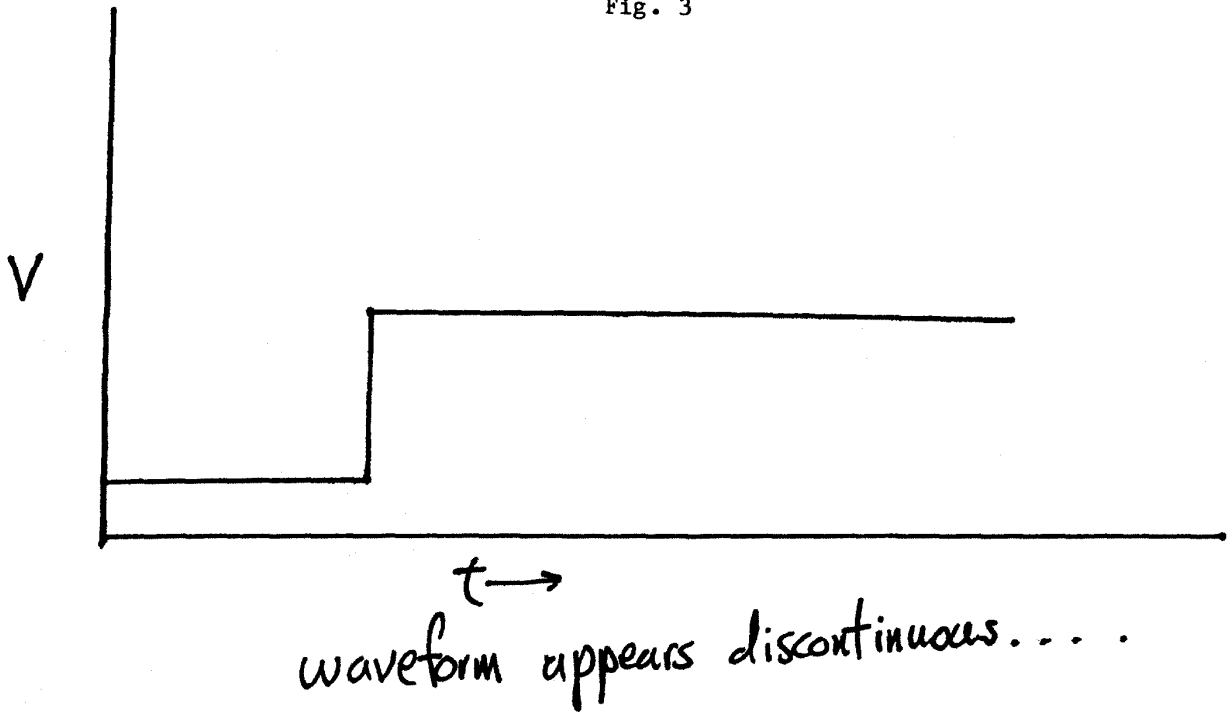


Figure 2. (a) circuit to initialize flip-flop shown with switches. To initialize device, switch R(run) is open and I is closed. To run device, reverse the situation. (b) transistor switched model version of (2a) set = 0 for initializing, set = 1 for run.

(c) There is no explicit time dependence in F . This means, roughly, that the system is driven by the voltages at any given time t and not by the time at which they occur. (d) There are two stable states at $(V_0, 0)$ and $(0, V_0)$. These are the "storage states" of the system and are the configurations which can retain information. We assume the function f is an infinitely differentiable function of V_A and V_B . This may seem like a severe restriction since we often represent physical quantities by non - or somewhat differentiable functions, like δ functions, step functions, functions with discontinuous derivatives, etc. However, these discontinuities are usually introduced as simplifications in models in which we can afford to overlook fine scale switching behavior due to the coarseness of our analysis. Until we reach a scale at which things become truly discontinuous (and who knows where that is, if indeed it is at all) we can invariably "find the smoothness in our step functions" by taking a finer look at the phenomena involved: e.g., a waveform which represents a voltage pulse may appear to be square--a useful approximation--until we take a closer look and see the "smooth edges" of the waveform (see Fig. 3). In any known situation where the concept of "electric circuit" is meaningful we are so far above the range (many orders of magnitude) where the question of the ultimate continuity of nature is important that it is safe to assume we can smooth our functions by refining the scale of observation. In any case, the general theory to be developed in this thesis can dispense with this differentiability criterion in many instances.

Fig. 3



until a closer look shows small changes
in signal

We have now proposed a model for a general bistable storage element implemented electrically with voltages in the range $[0, V_0]$. Let us pause to examine this model to see how well it corresponds to our notions of what a bistable device should be. It certainly has two stable states, by hypothesis. Perhaps the most interesting thing about this model is its generality. We have made absolutely no assumptions about the driving mechanism or the internal electronics or physics of the device. Indeed, it would be hard to imagine a bistable electrical device (or mechanical device, or thermal device - our model at no point uses the specific physical laws of electricity and magnetism) which could not be fit into the framework of this model. As far as I can tell, there are only two objections one might raise to this model. These will be handled separately.

The first objection might be that the model is too restrictive because it does not include an explicit dependence of the vector field F on t - i.e., F is not of the form $F(V_A, V_B, t)$. However, there are a large number of electrical devices for which $\partial F / \partial t = 0$; furthermore, an explicit time dependence in the driving vector field would in most practical circumstances be induced by external fluctuations such as temperature or power supply noise, or by some capacitance in the system which stores charge and hence contributes to the system in a history-dependent way. But the usual behaviour of a capacitor is given by

$$V = \frac{q}{C} = \int_{t_0}^t \frac{1}{C} dt .$$

This expression certainly looks as if it has specific time-dependence. But if we examine the rate of change of voltage with time, we see that

$$\frac{dV}{dt} = \frac{\dot{q}}{C} = \frac{1}{C}$$

which has no explicit time-dependence. Thus it seems reasonable, in the case of this simple example, to assume no explicit time-dependence for F . Furthermore, we shall see later that unless the system is indeed very badly behaved in its explicit time-dependence, the results will be independent of explicit time dependencies.

The second thing one might object to is the fact that we have made no real provision in our model as yet for the dynamics of external inputs. We have instead introduced external control in an ad hoc fashion by simply assuming we can move a point in the phase space to any other point at will. This is effectively what the switches and transistors in Fig. 2 accomplish, but at the cost of a drastic restructuring of the system in question. In fact, we are effectively turning off F while we initialize the system! This method of modeling inputs is acceptable as long as the time required for F to "turn on" after the switches are thrown is negligible. For typical TTL parts operating at nanosecond response times, this is an assumption of dubious worth. But the theory as developed in this thesis will, in fact, allow us to model very complex input responses in a framework similar to that we have set up here; furthermore, for many realistic models the assumptions used here are just fine. So it seems we have adopted a fairly general model for introductory purposes, one in fact which models a large number of bistable devices well.

It may then come as somewhat of a shock that any system satisfying properties a-d plus some slight additional requirements will (1) have other equilibrium points besides the two postulated; (2) these equilibrium points will not be "stable" in the sense that all trajectories starting sufficiently close to them asymptotically approach them as $t \rightarrow \infty$, as must be true of points $(0, V_0)$ and $(V_0, 0)$ if they are indeed stable points of the system, but (3) they will have some entering and some exiting trajectories; (4) for any time T , no matter how large, there is a finite probability that the system will not have reached a recognizable stable equilibrium by time T - if by "recognizing a stable equilibrium" we mean being within some finite prespecified distance of one in the $V_A - V_B$ plane.

These facts are proved using some elementary but deep results of differential topology, the main one of which is the Poincare-Hopf theorem on vector fields. This theorem associates with each isolated⁽³⁾ zero of a vector field V a "winding number"⁽⁴⁾ much as is used in the complex analysis: since we are dealing with an isolated zero, there is a small circle around the point where V is 0 (call it p) within which V has exactly one zero, at p , and on which it has no zeroes. On this circle, form the vector field $V/|V|$, a unit field. We can do this because $V \neq 0$ on C_p , the circle in question (Fig. 4). Then this new field gives a map from C_p to S^1 , the unit circle in the complex plane.

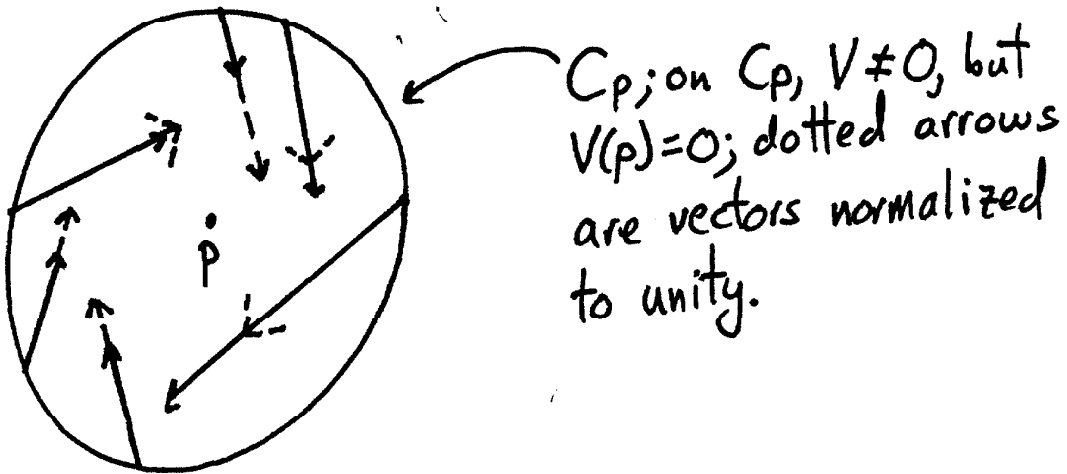


Fig. 4

The winding number of this map is the winding number of V at p . It is easily seen that the winding number of a vector field which runs into a point (which is the situation for stable equilibria) is $+1$. The Poincare-Hopf theorem on vector fields asserts that the sum of the winding numbers of a vector field about its zeroes is a number which is determined solely by the topology of the space in question, the Euler-Poincare characteristic. For a square, this number is one. Since we have two attracting points with winding number $+1$, and the Euler-Poincare characteristic of our space is 1, the theorem implies the existence of a point, a zero of the vector field, with winding number -1 . One can easily convince oneself that the kinds of zeroes with winding number -1 are "saddle point" zeroes--ones with both incoming and outgoing flow lines (integral curves of V). It is these points which give rise to metastable behavior, and their existence is unavoidable. This is one of the main points of this thesis. Statement (4) above is proved by noting that if we are on an integral curve approaching p , we never reach p , but approach it as

$t \rightarrow \infty$ and slow down in the process to zero speed. Now flow lines entering a small region of a "saddle point" either approach it asymptotically or leave the region eventually. We are interested in the ones which leave. Since by making them pass close enough to p we can slow them down as much as we please, we can, for certain flow lines, delay their entering a known region of convergence to a stable point as much as we want. This argument will be made more rigorous in Chapters 2 and 3 of this thesis.

These ideas were inspired by some work of Marino⁽²¹⁾ who proved similar results in a more general context. By specifying to the case of differentiable vector fields, we are able to obtain much more precise results.

The plan of attack is as follows: Chapter 2 covers the necessary mathematical material, and in Chapter 3 we return to analyzing this and other more complex devices from this point of view, but more thoroughly than we have here.

CHAPTER 2

TOPOLOGY AND GEOMETRY OF VECTOR FIELDS

1. INTRODUCTION

The purpose of this chapter is to give a clear, understandable account of the basic properties of vector fields on compact spaces (a term to be made clear later) at a level deep enough to be usable in conducting a rigorous discussion of metastable behaviour. We do not want to spend a lot of time proving what are basically support theorems, yet we do not want to leave out any important facts. Therefore, the following policy will be adopted: we will give all definitions and theorems in real "mathematician's language" but we will only give proofs of the more important ones, leaving the rest of the proofs to the references. In many instances, instead of proofs we will give explanations intended to clarify the subject matter.

2. BASIC MANIFOLDS; DEFINITIONS:

Before we can discuss vector fields and their properties in a coherent, useful way, we need to get some terminology down. This involves setting up some mathematical laboratory apparatus, so to speak. That will be our immediate concern. After that we will "run the equipment," and see what it can do.

The spaces we shall be concerned with are in the family known as the manifolds--and they are many and strange, indeed. Their basic claim to fame, what makes them useful, is that they look like Euclidean space in the small. This means near any point we can put coordinates on them, provided we do not try to make the coordinate system stretch "too far." Too far here means that the global properties of the manifold may not permit every point to fit in one Euclidean coordinate system by forcing one system to fold back on, bend, pinch or cut itself somehow. Think of a sphere or torus in Euclidean 3-space, for example. On the sphere, if we try to stretch a north polar coordinate system to cover the south pole we run into trouble. By moving the points around on the sphere, one can convince oneself of the following property for the sphere: given any two points, there is a coordinate system containing one of them and every point on the sphere--except the other specified point. So a manifold, or a space suitable for discussing the kinds of things we like to do with coordinates, need only have them extend locally around each point, but each point must have them.

Once we have coordinates near any point, we can talk about differentiable functions on the manifold. We do this by referring functions back to the coordinates. So if we have manifold M and a function f on it we say that f is differentiable at a point p if and only if f is differentiable in terms of the coordinates at p . There is one caveat here, however. There may be cases where a function is differentiable with respect to one coordinate system about a point but not another. To deal with this we have to restrict the coordinate systems at every point by placing a condition on the way they overlap. The condition is that when two coordinate system overlap, the

coordinates of one have to be differentiable functions of the coordinates in the other one. We now proceed with the formal statement of the requisite definitions.

We assume the reader is familiar with elementary topology in Euclidean space--in particular, we assume the basic properties of open sets are known.

Definition 1: R^n is n-dimensional Euclidean space.

Definition 2: Let f be a function from an open set U in R^n to R^m . Then f can be thought of as m n -argument real valued functions on U ,

$$f = (y^i(x)) \quad 1 \leq i \leq m$$

where

$$x = (x^1, \dots, x^n) \in U.$$

We say f is k times differentiable if for each y^i , all derivatives of y^i up to order k exist and are continuous. We use the notation $C^k(U, R^m)$ for the set of these functions. When the range and domain are clear from the context, we abbreviate this notation and say $f \in C^k$.

C^0 is the set of continuous functions.

C^∞ is the set of infinitely differentiable functions

C^ω is the set of real analytic functions, that is, functions which can be expanded in a power series about each point in U .

Clearly,

$$C^{\omega} \subseteq C^{\infty} \subseteq C^{1+1} \subseteq C^1 \quad i \in \mathbb{Z}^+ = \text{nonnegative integers}$$

Example: let $U = (-1,1) \subseteq \mathbb{R}^1$ and $f:U \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= e^{-x^{-2}} & x \neq 0 \\ &= 0 & x = 0. \end{aligned}$$

Then $f \in C^{\infty}(U)$ but $f \notin C^{\omega}(U)$ as can be seen by computing its Taylor series about 0: $d^k f/dx^k (0) = 0$ but f is nonzero in $(-1,1)$.

Definition 3: Let $U \subseteq \mathbb{R}^n$ be open and $f \in C^k(U, \mathbb{R}^m)$, $k \geq 1$. We define the derivative map of f as follows: Let $\mathbb{R}^{p \times q}$ denote the real vector space of $p \times q$ dimensional real matrices (p rows by q columns). The derivative of f , Df , is the map in $C^k(U, \mathbb{R}^{m \times n})$ given by associating with each point $x \in U$ the $m \times n$ matrix

$$\begin{bmatrix} \frac{\partial y^i}{\partial x^j} \end{bmatrix} \quad \begin{aligned} 1 \leq i \leq m, \\ 1 \leq j \leq n. \end{aligned}$$

If we consider any y^i , then its gradient as a function on U is the vector $(\partial y^i / \partial x^j)$. Thus Df has as its rows the gradients of the coordinates in the image space considered on functions on the range space.

We are now going to give a rather funny definition of vectors and vector fields on an open set U in \mathbb{R}^n ; while these concepts may seem strange at first, they will turn out to be the most logically consistent way to define things in the long run.

Suppose we have an open set $U \subseteq \mathbb{R}^n$ and a curve $\gamma(t) = (x^i(t))$, $1 \leq i \leq n$, given in parametric form. Let $f: U \rightarrow \mathbb{R}$ be a real-valued function on U , $f \in C^1$. Then $f \circ \gamma$ is a function on $I \subseteq \mathbb{R}^1$, the domain of γ . We can now compute the derivative of f with respect to t along γ (assuming $\gamma \in C^1(I, U)$). This is denoted by

$$\left(\frac{\partial f}{\partial t} \right)_{\gamma}$$

By the chain rule this is

$$(1) \quad \sum_i \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} \quad \text{where } \gamma = (x^i(t)).$$

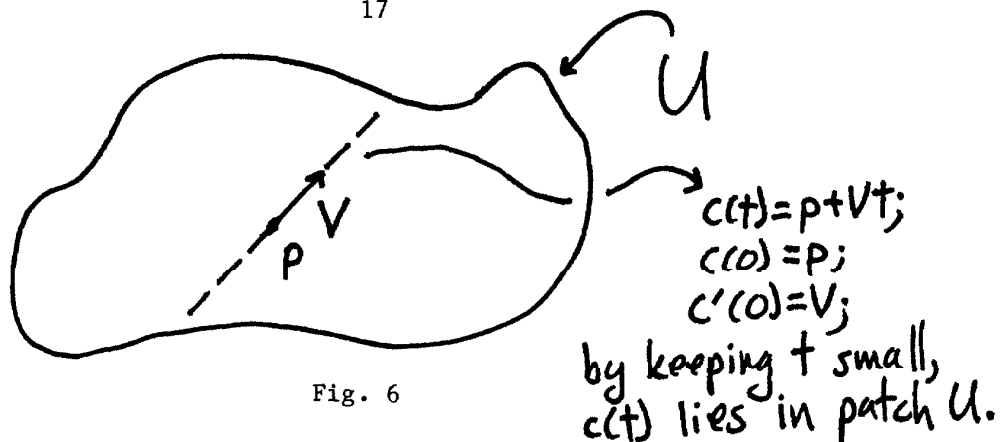
This is also equal to the directional derivative of f in the direction of the vector $(\dot{x}^i(t))$ at the point $(x^i(t))$. Now we can consider the vector $(\dot{x}^i(t))$ as being the differential operator on functions given by (1). From now on we shall identify tangent vectors to curves in U with differential operators in precisely this manner. Note that if we are at a point p in U and we have a vector V "at p " (see Fig. 6), we can form a curve $C(t)$ with tangent vector V by

$$C(t) = p + Vt;$$

$$\text{then} \quad C(0) = p$$

and

$$C'(0) = V.$$



(We restrict the range of t so that $C(t)$ actually lies in U .) The differential operator associated with this curve takes t to

$$\sum v^i \frac{\partial f}{\partial x^i};$$

in other words, it is

$$(2) \quad \sum v^i \frac{\partial}{\partial x^i}.$$

This shows there is a differential operator of the form (2) associated with each vector at each point in U . Now consider two curves γ_1 and γ_2 which have the same tangent vector at some point; that is, suppose

$$\gamma_1(0) = \gamma_2(0) = p$$

$$\gamma_1'(0) = \gamma_2'(0) = v.$$

Then an easy calculation using the chain rule as in (1) shows that the differential operators associated with γ_1 and γ_2 at p are the same, so

that in fact, though we used particular curves to define these operators, only their tangent vectors at the point of interest influence the operators at that point. This justifies our identification of tangent vectors to curves in U with differential operators.

Note that the coordinate tangent vectors at any point, the vectors $\partial/\partial x^j$, form a basis of the space of all tangent vectors at p . This is because, for any differential operator $\partial/\partial t$ associated with a curve $\gamma = (x^i(t))$,

$$\frac{\partial}{\partial t} = \sum \frac{dx^i}{dt} \frac{\partial}{\partial x^i}.$$

Now let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Let $f: U \rightarrow V$ be C^k , $k \geq 1$. Let $\gamma: I \rightarrow U$ be a C^k , $k \geq 1$, curve in U . Here we can take $I = (-1, 1)$. Then $f \circ \gamma$ is a curve in V . If $p \in U$ and $\gamma(0) = p$, $f(\gamma(0)) = f(p) \in V$. $(\partial/\partial t)_p$ is the tangent vector to γ at p . We define $f_*((\partial/\partial t)_p)$ to be the vector tangent to the curve $f \circ \gamma$ at $f(p)$. We can compute the action of $f_*((\partial/\partial t)_p)$ on g for any function $g: V \rightarrow \mathbb{R}$:

$$\begin{aligned} f_*\left(\left(\frac{\partial}{\partial t}\right)_p\right)g &= \left.\frac{dg}{dt}\right|_{f \circ \gamma} = \frac{dg(f \circ \gamma(t))}{dt} \\ &= \sum_i \frac{\partial g}{\partial y^i} \frac{dy^i}{dt} = \sum_i \frac{\partial g}{\partial y^i} \sum_j \frac{\partial y^i}{\partial x^j} \frac{dx^j}{dt} \end{aligned}$$

where $\gamma = (x^i(t))$. Thus the operator $f_*((\partial/\partial t)_p)$ at $f(p)$ in V is just $(Df) \left((\partial/\partial t)_p \right)$ where Df is the matrix of definition 3.

Now we introduce the following conventions: For any vector field X , (understood in the usual sense) X_p is the value at point p ; X_p also stands for some vector X at a point p .

Often we shall drop the p . Thus $f_*(X)$ is understood to mean f_* , computed at point p , acting on X , a vector at p .

Note that f_* only depends on X at p , not which $\gamma(t)$ gave rise to it. Thus f_* maps vectors to vectors. From now on all functions will be assumed to be as differentiable as needed. If a case arises where the degree of differentiability of a function is an issue, it will be noted.

Now f_* is given locally by matrix multiplication by Df . Suppose Df is of maximal rank as a linear transformation. Then since the space of tangent vectors to U at $p \in U$ has dimension n , if $m \geq n$ this rank is n . If $m < n$ it is m . Suppose $m = n$. Then Df has no kernel, i.e., the set of vectors X such that $Df(X) = 0$, $X \neq 0$, is empty. Thus Df is an isomorphism of vector spaces. This means (a) no tangent vector on U gets killed by f_* and (b) every tangent vector to v at $f(p)$ comes from some vector on U via f_* . So no curve γ in U has its tangent lost by f folding U around or something similar (see Fig. 7). Since the tangent vectors are local approximations to curves and coordinate lines, we should think the conditions that $\ker Df = 0$, i.e., that f_* is one-one and onto, should allow us to invert f , i.e., find a function $g: V \rightarrow U$ such that $fg = 1_V$ and $gf = 1_U$, where 1_V and 1_U are the identity functions on U and V , respectively.

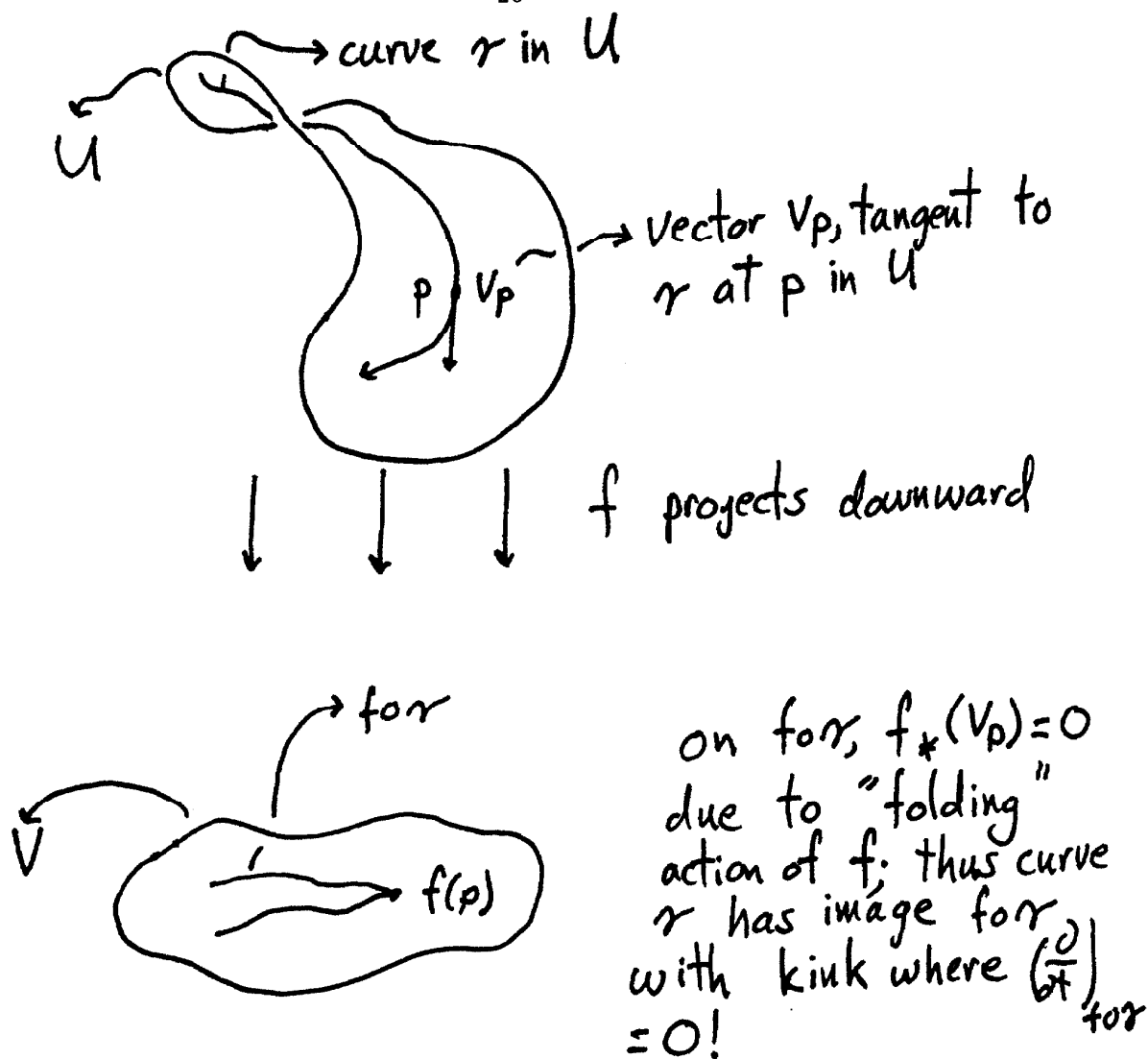


Fig. 7. How the action of f can lose vectors v_p by "folding".

That this is indeed the case locally is the content of the Inverse Function Theorem: Let $U \subseteq \mathbb{R}^n$ be open, let $f: U \rightarrow \mathbb{R}^n$ be C^∞ . Let $p \in U$ and suppose Df is nonsingular at p . Then there is an open set $V \subseteq U$, $p \in V$, and $f|V$ maps one-one onto the open set $f(V)$, and $f|V$ has a C^∞ inverse on $f(V)$.

This theorem was taken from the statement in Warner⁽⁵⁾. We have modified his notation slightly to suit our purposes. $f|V$ denotes f

restricted to V . For proof the reader should see the references given in (5). A slightly more advanced, but perhaps a conceptually cleaner proof, is given in Lang⁽⁶⁾.

A good way to understand this theorem is to visualize it in the one-dimensional case. Then Df is just f' and $f_*(V) = f'(p)V$. The theorem says if $Df(p) \neq 0$ then f can be inverted. Or, if $f'(x) \neq 0$, f is not parallel to the y axis (at least near p), so can locally be inverted (Fig. 8).

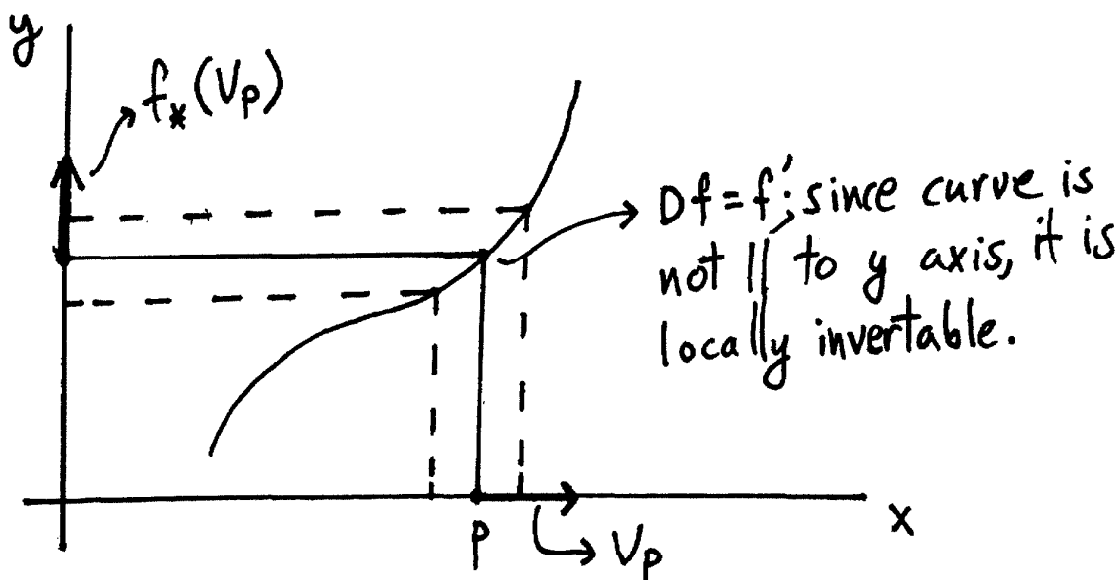


Fig. 8

We are now in a position to give a precise definition of a manifold. We assume familiarity with the basics of point-set topology. A reader needing information on this subject can consult (7). In particular, we assume familiarity with the concepts of open set, topology on a space, continuous map and homeomorphism.

Definition 4: Let M be a topological space. Suppose M is Hausdorff. M is said to be locally Euclidean of dimension d if for each point $p \in M$ there is an open set U with $p \in U$ and a homeomorphism $\phi: U \rightarrow V$, where V is an open subset of \mathbb{R}^d . We call the pair (U, ϕ) a chart at p . (Note for any $q \in U$, (U, ϕ) is also a chart at q .)

Definition 5: Let M be as above. A C^k atlas for M is a set of charts $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$, A an index set, such that

- (a) $\bigcup_{\alpha \in A} U_\alpha = M$
 (b) if $\alpha, \beta \in A$ then the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

is one-one, onto, C^k and has a C^k inverse.

Definition 6: A C^k manifold is a locally Euclidean space M together with a C^k atlas for M .

Note that our problem concerning functions differentiable in one coordinate system but not another has disappeared, since we have demanded coordinate systems have differentiable overlaps in (5b).

Note also that the map $\phi_\beta \circ \phi_\alpha^{-1}$ from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$ has nonsingular derivatives at all points. This follows from the following

Proposition 1. Let $f: U \rightarrow V$ and $g: V \rightarrow U$ be such that $f \circ g = 1_V$ and $g \circ f = 1_U$, U, V open in \mathbb{R}^n . Then at each point, Df and Dg are nonsingular matrices.

Proof: We need another

Lemma 1: If $f:U \rightarrow V$ and $g:V \rightarrow W$, U open in \mathbb{R}^n , V open in \mathbb{R}^m and W open in \mathbb{R}^p , then

$$(f \circ g)_* = f_* g_*$$

Proof: Work it out in coordinates using the chain rule. QED

Proof of proposition: $(Df)(Dg) = f_* g_* = (fg)_* = D(fg) = D(1_V) = 1$. Likewise $(Dg)(Df) = 1$. So Df and Dg , being inverses of each other, are nonsingular matrices. QED

It now follows that $\phi_\alpha \circ \phi_\beta^{-1}/\phi_\beta(U_\alpha \cap U_\beta)$ has nonsingular derivatives at all points, since $\phi_\beta \circ \phi_\alpha^{-1}/\phi_\alpha(U_\alpha \cap U_\beta)$ is an inverse of this map. This means that changing coordinates on a manifold must not only keep the points sorted out ($\phi_\alpha \circ \phi_\beta^{-1}$ is one-one and onto) it must keep tangent vectors sorted out—so it can not kink or otherwise mutilate curves in any topological sense.

This completes our discussion of what a manifold is and the elementary properties of manifolds. They will be the basic spaces about which the rest of this thesis will center.

3. VECTOR FIELDS AND FLOWS ON MANIFOLDS

We now proceed to generalize concepts we are familiar with from ordinary geometry in Euclidean space to manifolds in general.

Once again, we assume everything is as differentiable as we need to make the theorems and definitions work. Specifically, we will assume everything to be of class C^∞ unless otherwise stated.

Let M be a manifold of dimension n and $f:M \rightarrow \mathbb{R}$ be a real-valued function on M .

Definition 7: f is said to be differentiable on M if for each chart (U, ϕ) , $U \subseteq M$ open and $\phi:U \rightarrow \mathbb{R}^n$, the function $f \circ \phi^{-1}|_{\phi(U)}$ is differentiable on $\phi(U) \subseteq \mathbb{R}^n$.

So a function is differentiable iff it is differentiable in terms of a local coordinate chart. This is why we had to insist coordinate charts have differentiable transformation properties--so that differentiability of functions and vector fields will depend on the things and not the coordinates.

Recall from the previous section that a tangent vector at a point p is a differential operator

$$\sum v^i \frac{\partial}{\partial x^i}$$

which when applied to functions f gives the directional derivative of f in the direction of V at the point p .

Definition 8: A vector field on a manifold M is an assignment, to each point p of M , of a tangent vector at p . A vector field is differentiable iff in any chart (U, ϕ) the components V^i are differentiable.

Once we have the concept of vector field we are in a position to define differential equations and discuss their solutions. This we do next.

If U is an open set in \mathbb{R}^n , and $V(x)$ is a vector field on U , $x \in U$, then in the usual way we can consider V as defining a system of differential equations on U :

$$(3) \quad \dot{x}^i = \frac{dx^i}{dt} = V^i(x^1, \dots, x^n), \quad 1 \leq i \leq n.$$

where V^i is the i th component of V . We can consider a vector field as giving us a "velocity function" at each point which tells how that point will move with the time parameter t . More precisely, given any point $p \in U$, there is a curve $p(t) \subseteq U$ with $p(0) = p$ such that

$$\frac{dp^i}{dt} = V^i(p(t))$$

for all t for which p is defined. Tracing the point $p(t)$ as t changes gives what we can think of as the motion of a particle starting at p under the influence of the velocity vector field V . At time t , p has moved to $p(t)$. Thus we think of solutions to differential equations as curves $p(t)$; saying $p(0) = p$ gives an initial condition. The classical theory of ordinary differential equations guarantees a solution to (3) for any initial point and for all t as long as V is differentiable. An Excellent account of this can be found in (8).

The families of curves generated by a vector field in this manner have many interesting and useful properties. Consider, for example, the function $\phi_t(p)$ defined by

$\phi_t(p)$ = point of integral curve γ which satisfies $\gamma(0) = p$, at time t

= point which p gets moved to by velocity field V in time t .

Since one can prove that solutions to differential equations such as (3) are differentiable functions of the initial conditions as well as of t , the function ϕ_t is a differentiable function of p for any t . Furthermore, one can run t "backwards" as well as "forwards." This is tantamount to turning V around--replacing V by $-V$ at each point, and running t forwards. The net effect is that ϕ_{-t} undoes what ϕ_t does; that is

$$\phi_{-t}(\phi_t(p)) = \phi_t(\phi_{-t}(p)) = p.$$

Now since ϕ_t is differentiable and has a differentiable inverse, we can invoke proposition 1. Thus both ϕ_t and ϕ_{-t} have nonsingular derivatives at each point. This means that neither ϕ_t nor ϕ_{-t} tears, kinks or rips any curves--they only stretch and distort them.

Let $f:M \rightarrow N$ be a differentiable map between manifolds with a two sided differentiable inverse map $g:N \rightarrow M$.

Definition 9: We call such a map a diffeomorphism.

Obviously, diffeomorphisms preserve any differential property of a manifold, just as homeomorphisms preserve topological structure.

The above discussion indicates that each ϕ_t is a diffeomorphism. Actually, this is not quite the case as we have stated it. We still need to impose some mild restrictions on V . Furthermore, properly

speaking, we have only defined things for open sets U in Euclidean space. We need to extend the framework of the discussion to manifolds.

We take care of the second requirement first. Since a manifold looks like a bunch of open sets in Euclidean space, we can form local solutions to the differential equation (3) just as we do in Euclidean space. The overlap conditions on coordinate charts then assures us that the solutions will fit together properly. In this way we can extend the concept of integral curves of vector fields to manifolds.

The other obstacle we need to overcome concerns the way we have rather casually assumed that $\phi_t(p)$ is defined for all t and for all p in U . It may be in actuality that ϕ_t moves p out of U --then ϕ_t certainly is not a diffeomorphism of U to itself! However, a large body of mathematical literature has been devoted to showing that solutions exist and are well-behaved under just such circumstances. The interested reader can consult (6), Chapter IV. The problem is that the manifold may not have all the points we need to insure $\phi_t(p)$ is defined for all t --e.g., consider integral curves of the field $-\partial/\partial x$ in the x - y plane with the origin removed. If we start at $(x_1, 0)$, we cannot follow the integral curve to the t value x_1 --because the point which would be $\gamma(t)$ is not there for $t = x_1$!

Definition 10: We say a manifold M is complete for a vector field X if each integral curve of X can be extended from $t = -\infty$ to $t = +\infty$.

A compact manifold is always complete.

Since most of the manifolds we will deal with will be compact, this will not be a major issue.

When we need to explicitly stipulate completeness, we shall do so.

The function $\phi_t: M \rightarrow M$ or $\phi: \mathbb{R} \times M \rightarrow M$ is called the flow of the vector field. It is clear that a vector field determines a unique flow. Since a flow ϕ determines a vector field V by taking the tangent vectors of the integral curves, there is a one-to-one correspondence between vector fields and flows given explicitly by

$$V_p = \left. \frac{d}{dt} (\phi(t, p)) \right|_{t=0}$$

(here V_p means a tangent vector at the point p .)

So a flow ϕ_t can be thought of as describing the motion of an aggregate of particles under the influence of a vector field.

This concludes the section on vector fields and flows. We now move on to explore some properties of vector fields near certain special points of interest.

4. ZEROES OF VECTOR FIELDS AND THEIR INDICES

This section and the next are the most fundamental ones in this chapter in terms of their relevance to our later work. It is in these sections we shall begin to discuss the more interesting and subtle aspects of vector fields on manifolds; we shall especially be concerned with the behaviour of a vector field near a place where it has the value 0. It will turn out that it is these regions where "all the action is" for a vector field; furthermore, we shall discover there are certain fundamental relationships between vector fields near their zeroes and

the overall, or global, topology of the manifold as a whole. These relationships will be critical in our work to come.

First, we shall try to give some intuition as to why the zeroes of a vector field might be interesting in the first place. We begin by considering a region where a vector field has no zero.

Proposition 2: Let M be a manifold and V a vector field on M . Let $p \in M$ and suppose $V(p) \neq 0$. Then there is an open set U such that

- i) $p \in U$
- ii) $V(q) \neq 0$ for $q \in U$
- iii) U is diffeomorphic to an open cube in Euclidean space.

Definition 11: An open cube in a Euclidean space \mathbb{R}^n , centered at the point $r \in \mathbb{R}^n$, is a set of the form

$$\{y \in \mathbb{R}^n \mid |y^i - r^i| < s\}$$

for some positive number s . Here r^i stands for the i th component of r , as usual.

Proof of Proposition 2: Since M is a manifold, we know that there is an open set U , $p \in U$, and U is diffeomorphic to some open subset of Euclidean space. Thus we have a map

$$\phi: U \rightarrow E \subseteq \mathbb{R}^n.$$

Using this map we can transfer $V|_U$ to E via ϕ_* ; we call the new vector field V_E :

$$V_E(x) = \phi_*(V(\phi^{-1}(x))).$$

Now since $V(p) \neq 0$ and ϕ_* is nonsingular, (by proposition 1), $V_E(\phi(p)) \neq 0$. Then for some component of V_E^j of $V_E = \sum V_E^i \partial/\partial x^i$ (x^i coordinates on E , from the global Euclidean coordinates on \mathbb{R}^n), $V_E^j(p) \neq 0$. Since V_E^j is a continuous function (because differentiable) on E , there is an open ball in E , centered at $\phi(p)$, on which $V_E^j \neq 0$. Suppose this ball is $B(\phi(p), r)$. [We denote by $B(y, r)$ in \mathbb{R}^n the set

$$\{x \in \mathbb{R}^n \mid |y-x| < r\}$$

where $|z|$ is the Euclidean distance function

$$|z| = (\sum_1^2 z_i^2)^{1/2}.]$$

This ball contains an open cube of sufficiently small size centered at $\phi(p)$. Call it C . Then on $\phi^{-1}(C)$, $V \neq 0$ since $V_E^j \neq 0$ on the ball $B(\phi(p), r)$ containing C . $\phi^{-1}(C)$ is clearly diffeomorphic to an n -cube via $\phi|_{\phi^{-1}(C)}$. QED

Choosing the center of the cube in proposition 2 to be the origin (an easy adjustment to make), we see that proposition 2 gives a little Cartesian coordinate system with origin at p . Using this coordinate system we can examine V near p . (See Fig. 9). A little reflection shows that sufficiently near p the vector field V is doing generally what it is doing at p --it is pointing off in some direction, not too different than the direction at p . One can even show that by choosing coordinates appropriately, V can be made constant--in terms of the coordinates--near p . We sketch a proof of this.

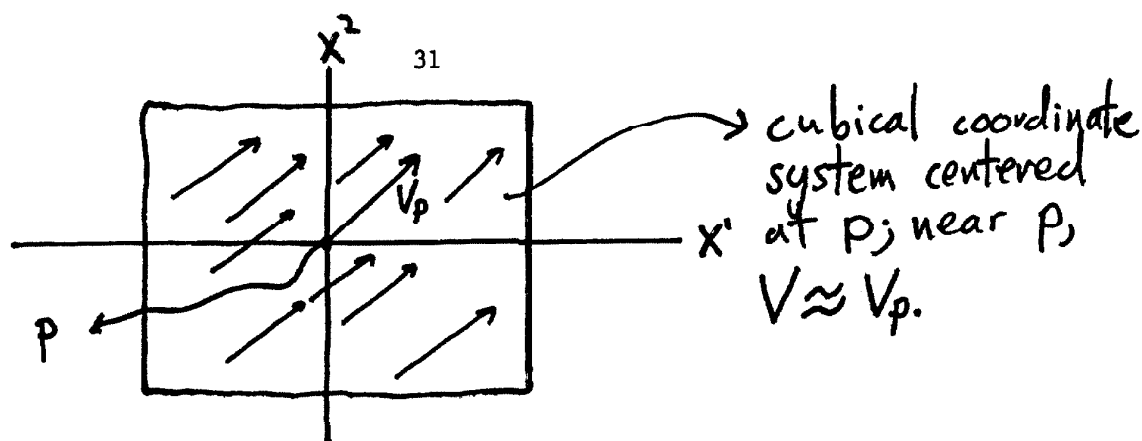


Figure 9. Vector field V near p ; $V_p \neq 0$

At p , V is some vector $\sum V^i \partial/\partial x^i$ -- or just the ordered n -tuple (V^1, \dots, V^n) , also denoted by (V^i) . Since $V \neq 0$, there is a subspace of the space of vectors at p (which looks like \mathbb{R}^n) which is normal to V . Call this subspace W . Near the origin, W consists of vectors of the form

$$\sum W^i \frac{\partial}{\partial x^i},$$

with the W^i small. In the coordinate system we have set up at p , consider the points whose coordinates are (W^i) . This is a little subspace of dimension $n-1$ and through which V passes. The important thing is V is not tangent to this subspace. Then the integral curves of V run through this subspace. Since the subspace (call it W') is of dimension $n-1$, we can put coordinates on it of the form y^1, \dots, y^{n-1} (by changing basis in the subspace W if need be). Then we use as coordinates near p y^1, \dots, y^{n-1} and t , the parameter of the integral curves of V , assigning a point q coordinates (y^1, \dots, y^{n-1}, t) iff $\phi_t(y^1, \dots, y^{n-1}) = q$, that is, q is the point (y^i) of the subspace W' moved along the integral curve of V passing through it by amount t . Since

$$V = \left(\frac{\partial}{\partial t} \right) ,$$

V has constant components in the (y^1, \dots, y^{n-1}, t) coordinate system. See

Fig. 10. A more detailed proof of this fact can be found in (6).

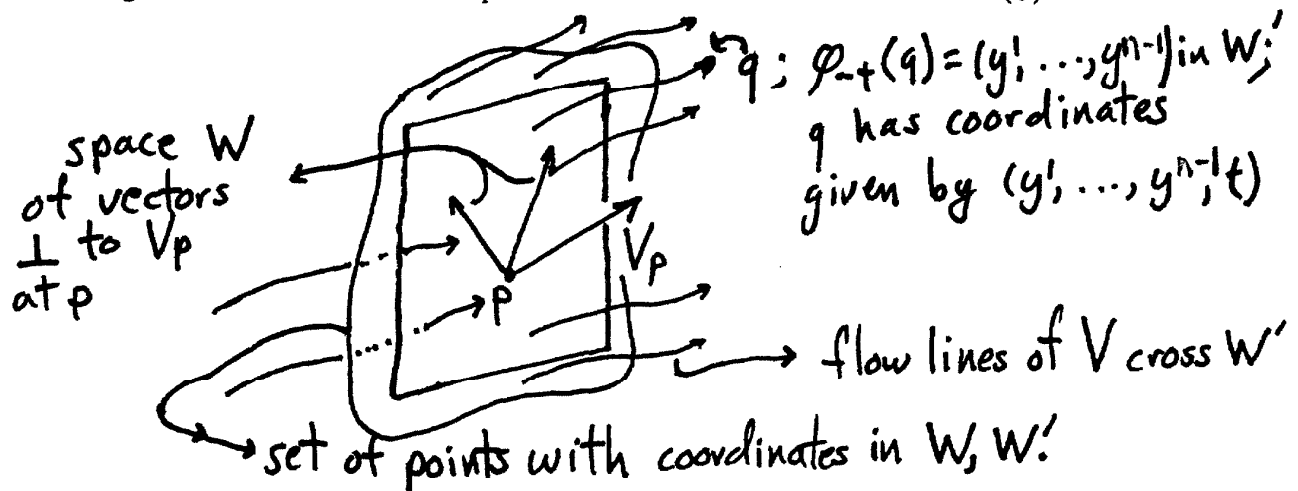


Fig. 10. A coordinate system in which a vector field is constant.
 W' gets coordinates from the space W at p .

The upshot of all this is that a vector field near a point where it is nonzero is rather uninteresting: it can even be made to look like a constant by choosing coordinates correctly. Nothing much goes on.

Now we consider the possibilities for vector field behavior in the neighborhood of a 0. For clarity, we confine ourselves to the two-dimensional case; we can generalize later.

Let us start out with a vector field which is nonzero at a point p and convert it to one which is zero via a function f such that

$$f(p) = 0$$

$$f(x) > 0 \quad x \neq p ,$$

assumed defined in some neighborhood U of p . Before and after snapshots of V and fV are given in Fig. 11.

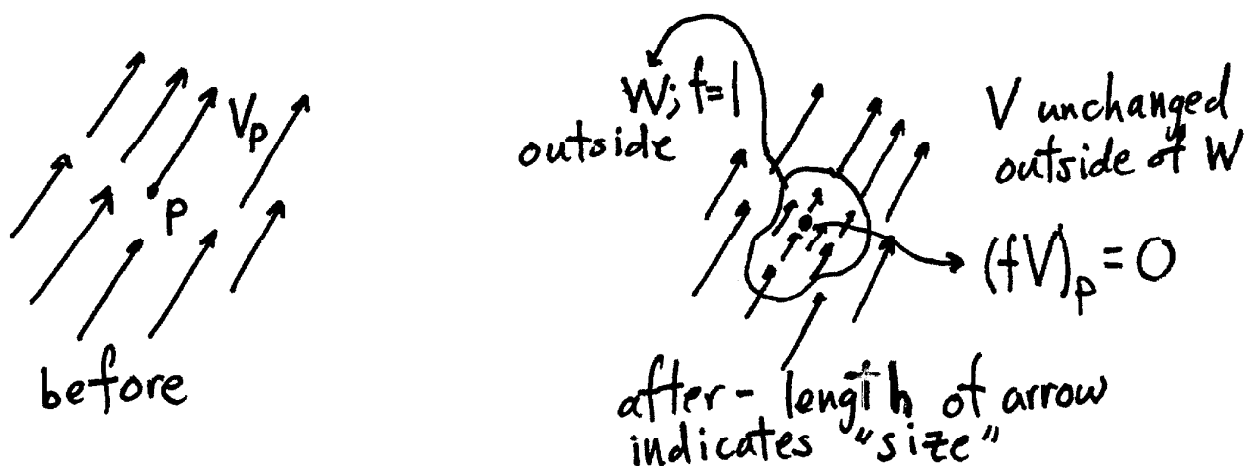


Figure 11

We observe the following: near p , but not at p , the vector field V has not been changed very much by multiplying it by the function f . Its size has changed, but it is still moving points along in much the same way as before. In fact, if we choose a neighborhood W of p we can choose f so that $f=1$ outside of W ; then V is unchanged outside of W . So all we have really done is to squish V down to zero in what can be an arbitrarily small neighborhood of p . The fact that V is zero at p does not affect its behavior away from p --in this case. We can push V down to zero at a point in this fashion without affecting it or its flow away from that point.

Now let us suppose V has a zero of a different flavor at p --one in which it comes in radially from all directions, like we would have if p were a stable equilibrium point of a gradient flow. (See Fig. 12).

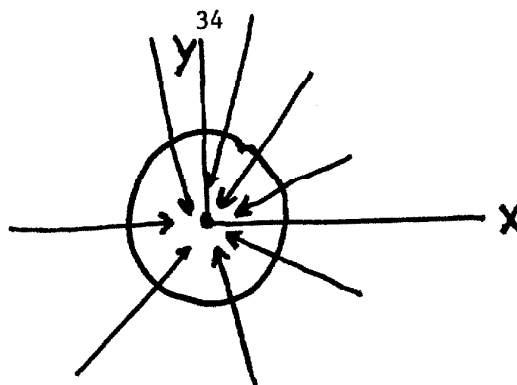


Fig. 12.

Vector field V coming in from all directions example:
 $V = -(x,y)$ in the plane

If we try to get rid of the zero in some simple manner we find we are stuck: we cannot alter the behavior of V at p without affecting all the curves, the flow lines of V , which are coming into p . In this case there is no way we can make V nonzero at or near p without doing violence to the global topological structure of the integral curves of V . The reason for this is, in a case like this one, V "wraps around" p . Since it comes in from all directions, p is somehow caught in the loop V makes as we go around p . We cannot simply redefine V at p , as we effectively did in the previous example, because that would give V a definite direction at p , hence near p , and V already has every possible direction near p . The zero at p is forced by the global behavior of V ; it is nontrivial in a topological sense.

These concepts are critical to the theory we are trying to develop. In order to make them more general and precise we need to introduce some basic concepts from algebraic topology. This we do in the following section.

5. HOMOLOGY AND THE INDICES OF VECTOR FIELDS

We now introduce the homology groups of a manifold. These are the most basic invariants in algebraic topology. It is in terms of them that the most important properties of vector fields will be stated. Though the definitions presented in this section may seem rather abstract, the homology groups are, in fact, a very straightforward way to measure the connectedness of a space. If one bears this in mind during the following discussion, the ideas introduced will be much easier to follow.

We begin by considering what is known as the standard p -simplex.

Definition 12: The standard p -simplex is defined to be the subset of \mathbb{R}^{p+1} given by

$$\{(x^i) \mid \sum x^i = 1 \quad \text{and} \quad \forall i, x^i \geq 0\} .$$

Thus the standard p -simplex is a generalized p -dimensional pyramid living in \mathbb{R}^{p+1} as the intersection of the hyperplane $\sum x^i = 1$ with the region where all coordinates have nonnegative values. The standard 0-simplex is the point $x=1$ in \mathbb{R} ; the standard 1-simplex is the line segment from $(0,1)$ to $(1,0)$ in \mathbb{R}^2 ; the standard 2-simplex is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in \mathbb{R}^3 , etc. These are illustrated in Fig. 13.

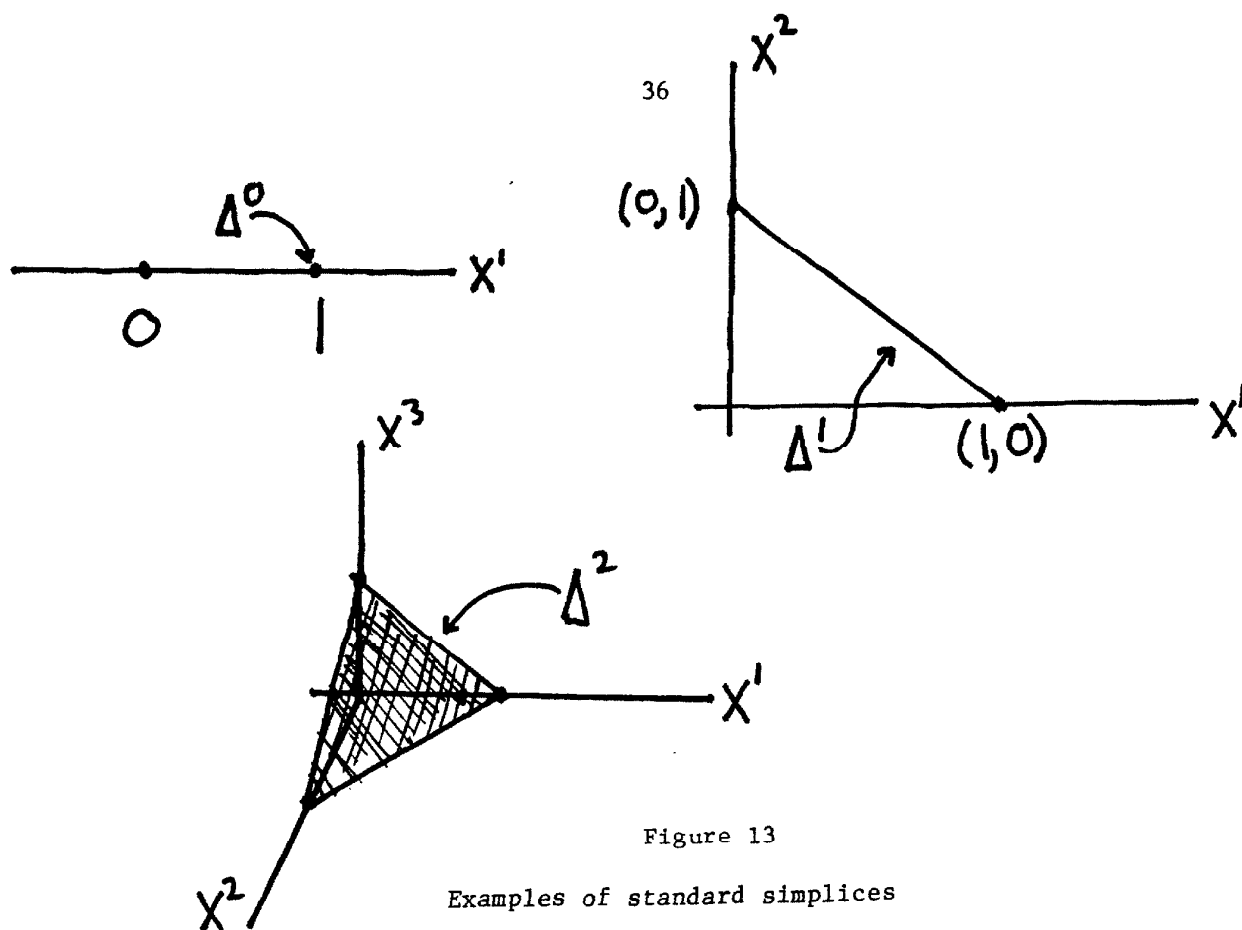


Figure 13

Examples of standard simplices

Now the critical thing to observe about p simplices is, except for the case $p=0$, the boundary of a p simplex consists of $p+1$ $p-1$ simplices. This can be seen by observing that the p simplex

$$\{(x^i) \mid \sum x^i = 1; \forall i, x^i \geq 0\}$$

intersects the $p+1$ planes $x^i=0$ each in a $p-1$ simplex

$$\{(x^j) \mid \sum x^j = 1; \forall j \neq i, x^j \geq 0, x^i = 0\}$$

which is the restriction of the hyperplane $\sum x^j = 1$ to the subspace where $x^i = 0$, all coordinates still assumed to be nonnegative.

If the coordinates in \mathbb{R}^{p+1} are assumed to be (x^0, \dots, x^p) , we have the following definitions:

Definition 13: Δ^p stands for the standard p -simplex.

Definition 14: The i th face of Δ^p is the $p-1$ simplex obtained by holding $x^i = 0$.

Now let M be an n -dimensional manifold.

Definition 15: A q -simplex in M is a function $\sigma: \Delta^q \rightarrow M$.

We sometimes informally identify a q -simplex in M , which properly speaking is a function from Δ^q to M , with its image in M , $\sigma(\Delta^q)$. So we can think of a q -simplex in M as a little q -dimensional pyramid drawn in M . See Fig. 14 for illustrations.

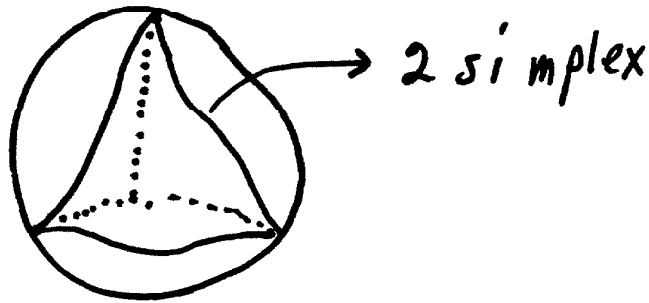


Figure 14. Simplices in the 2-sphere

We now form the q -simplices in M into a module⁽¹⁰⁾ over a ring S (assumed to be commutative with unit) by taking the finite formal linear sums

$$(\Lambda) \quad \sum r_i \sigma_i$$

where the σ_i are q -simplices in M , and each $r_i \in S$.

Definition 16: The set of all such sums is called the module of q chains in M with coefficients in S and is denoted by $C_q(M, S)$.

The default value of S in Definition 16 is Z , the ring of integers. When $S = Z$, we shall write simply $C_q(M)$.

Now let M be Δ^q itself. Then the faces Δ_i^q are in $C_{q-1}(\Delta^q, S)$ for any commutative ring with unit S . We now define the boundary of Δ^q , denoted by $\partial\Delta^q$, to be the element of $C_{q-1}(\Delta^q, S)$ given by

$$\partial\Delta^q = \sum_{i=0}^q (-1)^i \Delta_i^q ,$$

where, in S , $(-1)^i = 1$ if i is even and -1 if i is odd.

Having defined $\partial\Delta^q$ for the standard q -simplex we can extend the definition to q -chains in a manifold M by observing the following: a q simplex in M is a map $\sigma: \Delta^q \rightarrow M$. By restricting σ to Δ_i^q , we obtain a $q-1$ simplex. So we set

$$\partial\sigma = \sum (-1)^i \sigma(\Delta_i^q) .$$

We then extend ∂ to all of $C_q(M, S)$ by linearity. We obtain an S -linear map

$$\partial: C_q(M, S) \rightarrow C_{q-1}(M, S);$$

for $\alpha \in C_0(M, S)$, we set $\partial\alpha = 0$.

Definition 17: The map obtained by the above construction is called the boundary operator on $C_q(M, S)$.

We now need to deal with a minor technicality before we can make further progress. In defining ∂ on the standard q -simplex we referred to the faces Δ_i^q as if they were simplices in Δ^q . This is, of course, correct. Each Δ_i^q is topologically identical to a standard $(q-1)$ -simplex. But in order to be consistent with the notion that the boundary of a q -simplex is $(q-1)$ -chain, we need to specify just which map from Δ^{q-1} to Δ^q we mean by the face Δ_i^q . We want to do this in a consistent way; that is, when a face of a face occurs in Δ^q in two different simplices, the same map is used in both cases. If we define Δ_i^q to be the map f on Δ^{q-1} to Δ^q such that

$$(x^0, \dots, x^{q-1}) \mapsto (x^0, \dots, x^{i-1}, 0, x^i, \dots, x^{q-1})$$

it is easy to check that the desired condition holds. Furthermore, this choice of simplex in Δ^q for Δ_i^q gives the following useful and intuitively correct property:

Proposition 3: $\partial^2 = 0$.

Proof: Easily verified from the definitions. QED

We now single out some special chains.

Definition 18: We define $B_q(M, S)$ to be the set of q chains which are boundaries of $q+1$ chains:

$$B_q(M, S) = \partial C_{q+1}(M, S)$$

The elements of $B_q(M, S)$ are called the q -boundaries.

Definition 19: The set of cycles is the kernel of $\partial: C_q(M, S) \rightarrow C_{q-1}(M, S)$ and is denoted by $Z_q(M, S)$.

Note we set $C_q(M, S) = 0$ for $q < 0$.

Proposition 4: $B_q(M, S) \subseteq Z_q(M, S)$

Proof: Since $\partial^2 = 0$.

Since $B_q(M, S) \subseteq Z_q(M, S)$, we can form a precise measure of just how much bigger $Z_q(M, S)$ is than $B_q(M, S)$. We do this in

Definition 20: The q th homology group of M with coefficients in S is defined by the equation

$$H_q(M, S) = \frac{Z_q(M, S)}{B_q(M, S)} .$$

Thus $H_q(M, S)$ accounts for cycles which do not bound. We now consider the topological content of all this. Let us assume for the moment that $S = \mathbb{Z}$ = ring of integers (\mathbb{Z} is not to be confused with $Z_q(M, S)$!). In this case it is fairly obvious what the coefficients in the formal sums stand for: they count the redundancy with which a given simplex occurs in a chain.

Consider Proposition 3. Its meaning can be read almost literally from the equation: a boundary has no boundary. This corresponds nicely with our intuition; indeed, for any simple (and many not-so-simple) geometric figure which divide space into two parts, the boundary in the usual sense has no "edges," i.e., places where it stops and we cannot

cross into another part of the boundary. Think of the circle in the plane, or the sphere or torus in R^3 for example.

So boundaries are cycles. But they are rather trivial cycles. The reason is that they can be shrunk back rather severely through the things of which they are the boundary. Think of Δ^3 . Its boundary is composed of four simplices--the faces of a solid tetrahedron. This boundary clearly has the topology of a sphere, but since Δ^3 has no "holes" in it, the boundary can be pulled back into a single one of its points through Δ^3 . A little reflection will reveal that a similar thing must occur for any cycle which is a boundary. It may not be capable of being shrunk to a point, but it will always be capable of being distorted into something of lesser dimension. For this reason cycles which bound are not very significant topologically. Now a cycle which does not bound will not have this option. A good example can be had by taking Δ^3 of the above example and removing its interior--i.e., all points of the set

$$\{(x^i) \in \Delta^3 \mid \forall x^i \neq 0\} .$$

Then we are left with something which is topologically a 2-sphere. It is the sum of four simplices--the sides of a tetrahedron. It is clearly a cycle, but it does not bound anything. This corresponds precisely to the fact that a 2-sphere cannot be contracted to a point!

So the module $H_q(M)$, the cycles which do not bound, measures in a precise way the number of q dimensional "holes" in M .

Similar conclusions hold for other coefficient rings; these coefficients are useful in some contexts but for our purposes taking $S = Z$ will usually suffice.

We now give some useful facts on the relationship between homology groups and functions between manifolds. Let $f:M \rightarrow N$ be a function. For any q -simplex σ in M :

$$\sigma:\Delta^q \rightarrow M ,$$

we can form a q -simplex in N via functional composition

$$f \circ \sigma : \Delta^q \xrightarrow{\sigma} M \xrightarrow{f} N .$$

This map on simplices extends to chains by linearity: we call the map on chains $f_{\#}$. If $\alpha \in C_q(M, S)$ then

$$\alpha = \sum r_i \sigma_i ,$$

so

$$f_{\#}\alpha = \sum r_i f_{\#}\sigma_i = \sum r_i f \circ \sigma_i .$$

The following proposition is trivial.

Proposition 5: $f_{\#}$ commutes with ∂ . That is, if $f:M \rightarrow N$ and

$$\partial : C_q(M, S) \rightarrow C_{q-1}(M, S) ,$$

$$\partial : C_q(N, S) \rightarrow C_{q-1}(N, S) ,$$

then $f\partial = \partial f$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
 & \partial & \\
 C_q(M, S) & \xrightarrow{\quad} & C_{q-1}(M, S) \\
 \downarrow f_{\#} & & \downarrow f_{\#} \\
 C_q(N, S) & \xrightarrow{\quad \partial \quad} & C_{q-1}(N, S) .
 \end{array}$$

Proof: trivial check of definitions on simplices σ ; then extend by linearity.

The next proposition is equally easy to prove.

Proposition 6: $f_{\#}$ maps boundaries into boundaries and cycles into cycles.

Proof: Let $\alpha \in B_q(M, S)$. Then there is a β in $C_{q+1}(M, S)$ and $\partial\beta = \alpha$. Then $f_{\#}(\partial\beta) = f_{\#}(\alpha)$. $f_{\#}(\partial\beta) = \partial f_{\#}(\beta)$ so $f_{\#}(\alpha) \in B_q(N, S)$. If $\alpha \in Z_q(M, R)$ then $\partial\alpha = 0$ so $\partial f_{\#}(\alpha) = f_{\#}(\partial\alpha) = 0$. QED.

Thus $f_{\#}$ defines a map on $H_q(M, S) = B_q(M, S)/Z_q(M, S)$ to $H_q(N, S)$. This map is denoted by f_{\star} (not to be confused with the map f_{\star} or vectors). This also follows easily:

Proposition 7: if $f: M \rightarrow N$ and $g: N \rightarrow P$ are maps, then $(fg)_{\star} = f_{\star} \circ g_{\star}$.

Proof: trivial. QED.

Now the most important thing about the map f_* induced by f rests on the concept of homotopy; that is, the deforming of one map to another. This discussion begins with the

Definition 20: Let $f, g: M \rightarrow N$ be maps of spaces. A homotopy between f and g is a function

$$F : M \times I \rightarrow N ,$$

where I is the closed unit interval $= [0,1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$, with the following properties:

- (i) F is continuous
- (ii) $F(x,0) = f(x)$ for all $x \in M$
- (iii) $F(x,1) = g(x)$ for all $x \in M$.

Thus a homotopy is a one-parameter family of maps; we can think of map f "deforming" to map g as t runs from 0 to 1. We say maps f and g are homotopic.

The key result in homotopic maps is

Theorem 1: If f and g are homotopic maps, written $f \simeq g$, then $f_* = g_* : H_q(M, S) \rightarrow H_q(N, S)$.

The proof of this is long, involved, technical and depends on much more material than we need. See (10).

An interesting and useful result is the following:

Proposition 8: Let ϕ_t be a flow on M . For any fixed value of t , ϕ_{t_0} is homotopic to the identity map M, I_M .

Proof: Since $\phi_0(x) = x$ for all $x \in M$, $\phi_0 = I_M$ for any flow ϕ_t . Then a homotopy of ϕ_{t_0} to I_M is given by

$$F(x, t) = \phi_{t-t_0}(x) . \quad \text{QED.}$$

We now give some examples of homotopy groups of some familiar spaces. We first give a result which typifies the kind of calculations we do.

Definition 21: Let X be a space and A a subspace of X . A is said to be a deformation retract, abbreviated d.r., of X iff there is a map $h: X \rightarrow A$ such that $h(a) = a$ for $a \in A$ and $L_A h$ is homotopic to 1_X where L_A is the inclusion map of A in X .

Example:

(1) The closed unit interval I has as a deformation retract any of its points, for if $p \in I$, the map $f: I \rightarrow \{p\}$ is homotopic to 1_I via

$$F(r, t) = p + (1-t)(r-p)$$

(2) The circle is a deformation retract of the punctured disk.

(3) The origin is a deformation retract of \mathbb{R}^n .

We leave construction of the homotopies in (2) and (3) to the reader.

The reader may also easily prove that homotopy is an equivalence relation, on maps, i.e., if we write $f \simeq g$ for f is homotopic to g , then $f \simeq f$, $f \simeq g$ iff $g \simeq f$ and $f \simeq g$ and $g \simeq h$ imply $f \simeq h$.

We have

Proposition 9: If A is a deformation retract of X , then the inclusion map $L_A: A \rightarrow X$ induces an isomorphism of homology groups

$$L_{A*} : H_q(A, S) \rightarrow H_q(X, S)$$

for all q .

Proof: If A is a d.r. of X , there is a map $h: X \rightarrow A$ with properties specified in Definition 21. Note that $hL_A = 1_A$ and $L_A h = 1_X$. Thus $h_* L_{A*} = 1_{A*}$ and $L_{A*} h_* = 1_{X*}$. Thus $L_{A*}: H_q(A, S) \rightarrow H_q(X, S)$ is invertible with inverse h_* . So it is an isomorphism. QED.

We denote the $H_q(X, S)$ collectively by $H_*(X, S)$, and say $f: X \rightarrow Y$ induces $f_*: H_*(X, S) \rightarrow H_*(Y, S)$.

Now to our examples:

We start with what must be one of the few truly evident facts in mathematics:

$$\text{for a point } p \quad H_0(p) = \mathbb{Z} ,$$

$$H_i(p) = 0 \quad \text{if } i \neq 0.$$

Now consider an interval J in \mathbb{R} : for any interval in \mathbb{R} and any point p in the interval, p is a deformation retract of J , thus

$$H_i(J) = \begin{array}{ll} \mathbb{Z} & \text{if } i = 0 \\ 0 & i \neq 0 . \end{array}$$

More generally, we have

Definition 22: A space X is said to be contractible if the map 1_X is homotopic to a constant map $X \rightarrow x_0 \in X$.

Proposition 10: A contractible space has the homology groups of a point.

Proof. Let $x_0 \in X$ be a point such that X is contractible to x_0 . Then 1_X is homotopic to the constant map $X \rightarrow x_0$. Call this map h . Clearly, $h(x_0) = x_0$. Consider $L_{x_0} h$. $L_{x_0} h(x) = L_{x_0}(x_0) = x_0 = h(x)$ for all $x \in X$, and $h(x) \simeq 1_X$, so $L_{x_0} h \simeq 1_X$. Thus x_0 is a deformation retract of X and hence has the same homology groups. QED.

Thus R^n , the unit ball in R^n , and intervals all have the homotopy groups of a point.

A circle S^1 has

$$H_0(S^1) = \mathbb{Z}$$

$$H_1(S^1) = \mathbb{Z}$$

$$H_i(S^1) = 0 \quad i \neq 0, 1.$$

$H_1(S^1) = \mathbb{Z}$ measures to "winding" number of any map $S^1 \rightarrow S^1$ (which corresponds to a cycle). Each $n \in \mathbb{Z}$ is the homology class of a map like $e^{i\phi} \rightarrow e^{in\phi}$, which wraps S^1 around itself n times.

Thus a punctured disk (of which the circle is a deformation retract) has the same homology groups as the circle.

For $S^1 \times S^1$ we have

$$H_0(S^1 \times S^1) = \mathbb{Z} ,$$

$$H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} ,$$

$$H_2(S^1 \times S^1) = \mathbb{Z}$$

$$H_i(S^1 \times S^1) = 0 \quad i \neq 0,1,2 .$$

" $H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$ " says there are two maps $v_1, v_2, S^1 \rightarrow S^1 \times S^1$, each going around different "holes." A moment's reflection shows this to be true.

For an n -sphere, S^n , if $n \neq 0$,

$$H_0(S^n) \approx \mathbb{Z}$$

$$H_n(S^n) \approx \mathbb{Z}$$

$$H_i(S^n) = 0 \quad \text{if} \quad i \neq 0, n .$$

" $H_n(S^n) \approx \mathbb{Z}$ " says there is a "generalized winding number" for a map $f: S^n \rightarrow S^n$ (which corresponds to a cycle) and that the map whose "winding number" is n wraps S^n around itself n times. This last notion ties us in to vector fields again. Let V be a vector field on a manifold M . Let $p \in M$ be an isolated 0 of V . This means there is a neighborhood U of p such that V has no zero in $U - p$. If we choose a cubical coordinate system centered at p we can find a sphere Σ in this coordinate system, centered at p , on which V has no zero. By rescaling coordinates we can make this sphere the set $\{(x^i) \mid \sum (x^i)^2 = 1\}$. If we examine V in this

coordinate system we see that it is given by an expression of the form

$$\sum v^j \frac{\partial}{\partial x^j}$$

where the v^j are functions of the x^i . Now since the vector (v^j) has no zero on Σ , so $\sum (v^j)^2 \neq 0$ on Σ . We can thus normalize v^j to unity on this sphere. Note this normalization affects only the size of V , not its direction. Thus it does not affect the number of times V winds around the 0 at p on Σ . Furthermore, since V is normalized on Σ , we can consider V_Σ as a map

$$\Theta_V : \Sigma \rightarrow \Sigma .$$

Now this map Θ_V induces a map

$$\Theta_{V*} : H_*(\Sigma) \rightarrow H_*(\Sigma) .$$

Since Σ has the topology of an $n-1$ dimensional sphere,

$$H_{n-1}(\Sigma) = \mathbb{Z} ,$$

so Θ_{V*} can be thought of as a homomorphism $\Theta_{V*} : \mathbb{Z} \rightarrow \mathbb{Z}$. What does a homomorphism of $\mathbb{Z} \rightarrow \mathbb{Z}$ look like? Since \mathbb{Z} is generated by 1, it is easy to compute $\Theta_{V*}(m)$ for any $m \in \mathbb{Z}$

$$\Theta_{V*}(m) = \Theta_{V*}(m \cdot 1) = m \Theta_{V*}(1) .$$

Thus Θ_{V*} is completely determined by $\Theta_{V*}(1)$.

Definition 23: The index of V at p is the integer $\theta_{V*}(1)$.

The index of a vector field at a point measures how fast the vector field is changing near the point, in a purely topological sense. It is a direct generalization of the concept of winding number used in complex analysis. Part of the use of the index of a vector field at a point stems from the following

Proposition 10: Let V_1 and V_2 be vector fields on a manifold M . Suppose $V_1(p) = 0$, $p \in M$, and p is an isolated zero of V_1 . Let Σ be a sphere about p in some coordinate patch as described above. Suppose V_2 is related to V_1 through a one-parameter family of vector fields; that is, a set of vector fields $V(t)$, parametrized in differentiable way by $t \in [0,1]$ such that $V(0) = V_1$ and $V(1) = V_2$. Suppose that for all $t \in [0,1]$ and $q \in \Sigma$, $V(t)(q) \neq 0$. Then the index of V_1 at p equals the index of V_2 at p .

Proof: The conditions of the above proposition imply that the maps from Σ to Σ we get are homotopic, hence $\theta_{V_1*} = \theta_{V_2*}$ on $H_*(\Sigma)$, and for $\theta_{V_1*} = \theta_{V_2*} : H_{n-1}(\Sigma) \rightarrow H_{n-1}(\Sigma)$, ($n = \dim M$) we have $\theta_{V_1*}(1) = \theta_{V_2*}(1)$. QED.

Proposition 10 serves more than one very valuable purpose: firstly, it allows us to conclude that if we change V_1 by a small amount, the index will not change.

Definition 24: We denote the index of V at a point p by $\theta_V(p)$. Thus

$$\theta_{V*}(1) = \theta_V(p)$$

for θ_{V*} computed at p .

Now why will not $\Theta_V(p)$ change? Changing V by a small enough amount will not move any zeroes across Σ so we obtain homotopic maps from $\Sigma \rightarrow \Sigma$ via the homotopy induced by $V(t)$, where $V(t)$ is the parametrized vector field.

Thus the index $\Theta_V(p)$ has a certain stability property: it will not change under small perturbations of V .

The second point of interest is that as long as no zero of V explicitly crosses Σ , we can do anything we want to V in the way of smoothly changing without affecting $\Theta_V(p)$. We can even disturb V in such a way that new zeroes are introduced on the inside of Σ . The interesting thing is that they will have to come into being in such a way as to keep the index constant; even if one of them increases $\Theta_{V*}(1)$, some other one will have to come into being in such a way as to keep $\Theta_{V*}(1)$ constant. This fact and similar lines of reasoning will be very helpful in understanding

5. FIXED POINTS AND THE GLOBAL PROPERTIES OF VECTOR FIELDS

We begin this section by considering general functions $f:M \rightarrow M$, M a compact manifold.

We know f induces a map f_* on $H_*(M)$. What does a typical $H_1(M)$ look like? Note that here we use integral coefficients. The compactness of M implies that for all i , $H_i(M)$ is a finitely generated module over the integers. Such a module can always be written as a direct sum of two submodules $F \oplus T$, where F is free, that is, is a direct sum of some finite number of copies of \mathbb{Z} and T is a torsion module, that

is, for any $t \in T$, there exists $m \in \mathbb{Z}$ such that $mt = 0$ and $m \neq 0$. Note for $f \in F$, $mf = 0$ with $m \neq 0$ iff $f = 0$. These facts are standard elementary algebra, as can be found in (11), pp. 7-9, pp. 193-199 or (12), pp. 179, 220.

Now consider $f_*: H_1(M) \rightarrow H_1(M)$. We call this map f_1 . If $H_1(M) = F_1 \oplus T_1$, we can consider f_1 restricted to $F_1 = \mathbb{Z}^{\alpha_1}$. We get a map from $F_1 \rightarrow F_1 \oplus T_1$, denoted by $f_1|_{F_1}$. We can project this map in turn into F_1 , obtaining a map, called $F(f_1)$, from $F_1 \rightarrow F_1$. This business is diagrammed below:

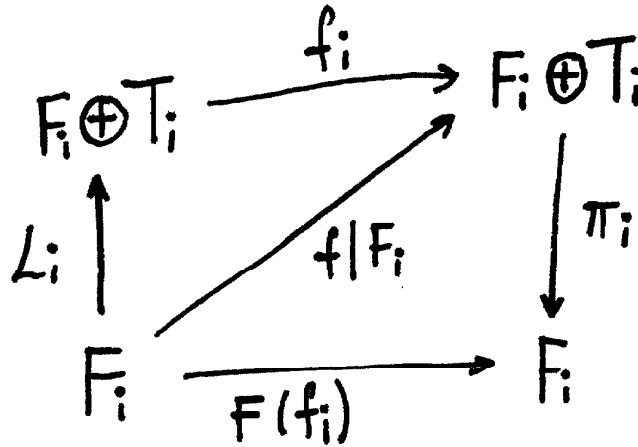


Fig. 15. Commutative diagram showing how $F(f_1)$ is defined

$L_1: F_1 \rightarrow F_1 \oplus T_1$ is the inclusion map; $\pi_1: F_1 \oplus T_1 \rightarrow F_1$ is the projection map

Definition 24: We define the trace of f_1 , denoted $\text{Tr} f_1$, to be the trace of the map $F(f_1)$.

Since $F_1 = \mathbb{Z}^{\alpha_1}$, and $F(f_1): F_1 \rightarrow F_1$, we can pick a basis e_j for F_1 , $1 \leq j \leq \alpha_1$. Then we have

$$F(f_i) e_j = \sum_{k=1}^{\alpha_i} r_{jk} e_k \in F_i.$$

Thus the matrix of $F(f_i)$ with respect to the basis $F(f_i)$ is (r_{jk}) and $\text{Tr}(f_i) = \sum_{j=1}^{\alpha_i} r_{jj}$. From its definition, $\text{Tr}(f_i)$ is invariant under homotopies of f . We now define another homotopy invariant quantity associated with f :

Definition 25: We define the Lefschetz number of a map $f : M \rightarrow M$, denoted by $\lambda(f)$ by the equation

$$\lambda(f) = \sum_{j=0}^n (-1)^j \text{Tr}(f_j),$$

where $n = \dim M$.

Definition 26: Denote by $\rho_i(M)$ the rank of $H_i(M) = F_i \oplus T_i$; then $\rho_i(M)$ = cardinality of a basis for F_i . The Euler-Poincare characteristic of M , denoted by $\chi(M)$, is defined to be

$$\sum_{i=0}^n (-1)^i \rho_i(M);$$

$\rho_i(M)$ is called the i th Betti number of M . Here $n = \dim M$.

$\chi(M)$ is also referred to as simply the Euler characteristic of M .

$\lambda(f)$ is a generalization of $\chi(M)$ as the following proposition and theorem show:

Proposition 11: Let $1_M : M \rightarrow M$ be the identity map. Then $\lambda(1_M) = \chi(M)$.

Proof: 1_M induces the identity map 1_{M*} on $H_*(M)$. The trace of 1_M is $\rho_i(M)$. The conclusion follows directly from this. QED.

Theorem 2: If $f : M \rightarrow M$ is homotopic to 1_M , then $\lambda(f) = \chi(M)$.

Proof: Proposition 11 and the fact that $f \simeq 1_M \rightarrow f_* = 1_{M*}$. QED.

The significance of the Lefschetz number of a map lies in the fact that it provides a direct link to the structure of the fixed point set of the map f , that is, the set of all $x \in M$ such that $f(x) = x$. This is stated explicitly in the following

Theorem 3: Let M be a compact manifold, and $f: M \rightarrow M$ be a map. If $\lambda(f) \neq 0$, there exists an $x \in M$ such that $f(x) = x$.

In order to sketch a proof of this theorem we need to develop some insight into the behavior of the Lefschetz number of a map. The most important fact is that $\lambda(f)$, though defined in terms of the action of f_* on the homology of M , can actually be defined in terms of the action of $f_\#$ on the $C_q(M)$ themselves, or at least on certain submodules of the $C_q(M)$. We now proceed to do this.

First of all, note that the $C_q(M)$, being generated by all maps $\sigma: \Delta^q \rightarrow M$, are in general infinite dimensional modules, that is, there is no finite set $C = \{C_1, \dots, C_r\} \subseteq C_q(M)$ such that for all $c \in C_q(M)$, $c = \sum n_i C_i$ for suitable integers n_i . The homology modules $H_q(M)$, on the other hand, are, for compact M , finite-dimensional. Thus $\text{Tr}(f_q)$ is well-defined on $H_q(M)$, but not necessarily on $C_q(M)$. In order to calculate $\lambda(f)$ via the $C_q(M)$, we need finite-dimensional submodules of the $C_q(M)$, say $\hat{C}_q(M)$, which contain all the topological (more precisely, homological) information the $C_q(M)$ do. In fact, such finite-dimensional submodules always exist; the proof of this fact is straightforward, long-winded, and technical and need not be given here. The interested reader can consult Chapter 4 of (10).

The basic reason the $\hat{C}_q(M)$ exist and have the requisite properties is that the $C_q(M)$ are extremely redundant in their expression of topological data; that is, up to homotopies (deformations) of chains, there are, for a compact manifold, really only a finite-dimensional set of essential chains. The critical result relating the Lefschetz number as computed on the $\hat{C}_q(M)$ and on $H_q(M)$ is

Theorem 4: Let the $\hat{C}_q(M)$ be finitely generated, and finite in number (i.e., $\hat{C}_q(M) = 0$ for all but a finite number of q). Let $f_{\#} : \hat{C}_q(M) \rightarrow \hat{C}_q(M)$ be a set of maps such that

$$f_{\#} \circ \partial = \partial \circ f_{\#} ,$$

and let f_{\star} be the map $f_{\#}$ induces on $H_{\star}(M)$. Then

$$\lambda(f_{\star}) = \lambda(f_{\#}) ,$$

where

$$\lambda(f_{\#}) = \sum (-1)^n \text{Tr} (f_{\#n}) ,$$

$f_{\#n}$ being the map $f_{\#}$ on $\hat{C}_n(M)$.

This theorem is proved using straightforward algebra. The interested reader can consult (10), pp. 172-173, where he will also find out why the alternating sum is used in defining $\lambda(f)$: the alternating sum gives rise to a certain cancellation property which makes Theorem 4 true.

Once we have Theorem 4, Theorem 3 is easy. Suppose $f:M \rightarrow M$ has no fixed points. Then since M is compact, we can put a metric on M such that f moves each point by at least some $\epsilon > 0$, i.e., $d(x, f(x)) > \epsilon$. Now look at the $\hat{C}_q(M)$. We can always choose the simplices in $\hat{C}_q(M)$ to be as small as we please. If they get smaller than $\epsilon/2$ in diameter, then the relation $d(x, f(x)) > \epsilon$ implies no simplex touches its image under f , which implies $\text{Tr}(f_{\#q}) = 0$ on all $\hat{C}_q(M)$, so by theorem 4 $\text{Tr}(f_q) = 0$ on all $H_q(M)$, hence $\lambda(f) = 0$. Note we have actually proved that, for all q , $\text{Tr}(f_q) = 0$ if f has no fixed point on M .

Another useful feature of $\lambda(f)$ is that it can be computed locally near the fixed points of f (if there are any). Unfortunately, a rigorous explanation of this lies beyond the scope of this work. The interested reader can consult (20). However, it is true that, for a vector field \vec{X} with isolated zeroes on a compact manifold M , the local index of ϕ_ϵ at a point p , where ϕ is the flow of \vec{X} and ϵ is sufficiently small, is just $\theta_{\vec{X}}(p)$. The theory cited tells us that

$$\lambda(\phi_\epsilon) = \text{sum of local indices,}$$

so

$$\lambda(\phi_\epsilon) = \sum_{\substack{\vec{X} \\ p \text{ a zero of } \vec{X}}} \theta_{\vec{X}}(p)$$

Since $\phi_\epsilon \simeq 1_M$, $\lambda(\phi_\epsilon) = \chi(M)$, so we obtain the Poincare-Hopf Theorem:

$$\sum_{\substack{\vec{X} \\ p \text{ a zero of } \vec{X}}} \theta_{\vec{X}}(p) = \chi(M)$$

Another useful fact is, if $\vec{X}(p) = 0$, and $D\vec{X}(p)$ is nonsingular, $\text{sign}(\det(D\vec{X}(p))) = \theta_{\vec{X}}(p)$. This can be proved by computing the index

for a nonsingular linear vector field explicitly, and then showing that \vec{X} near p is homotopic to the linear field given by the derivative (i.e., $D\vec{X}(p)$).

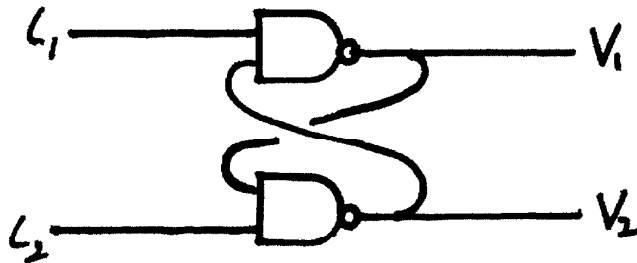
This concludes the mathematical preliminaries we shall need. We move on to analyzing particular devices.

CHAPTER 3

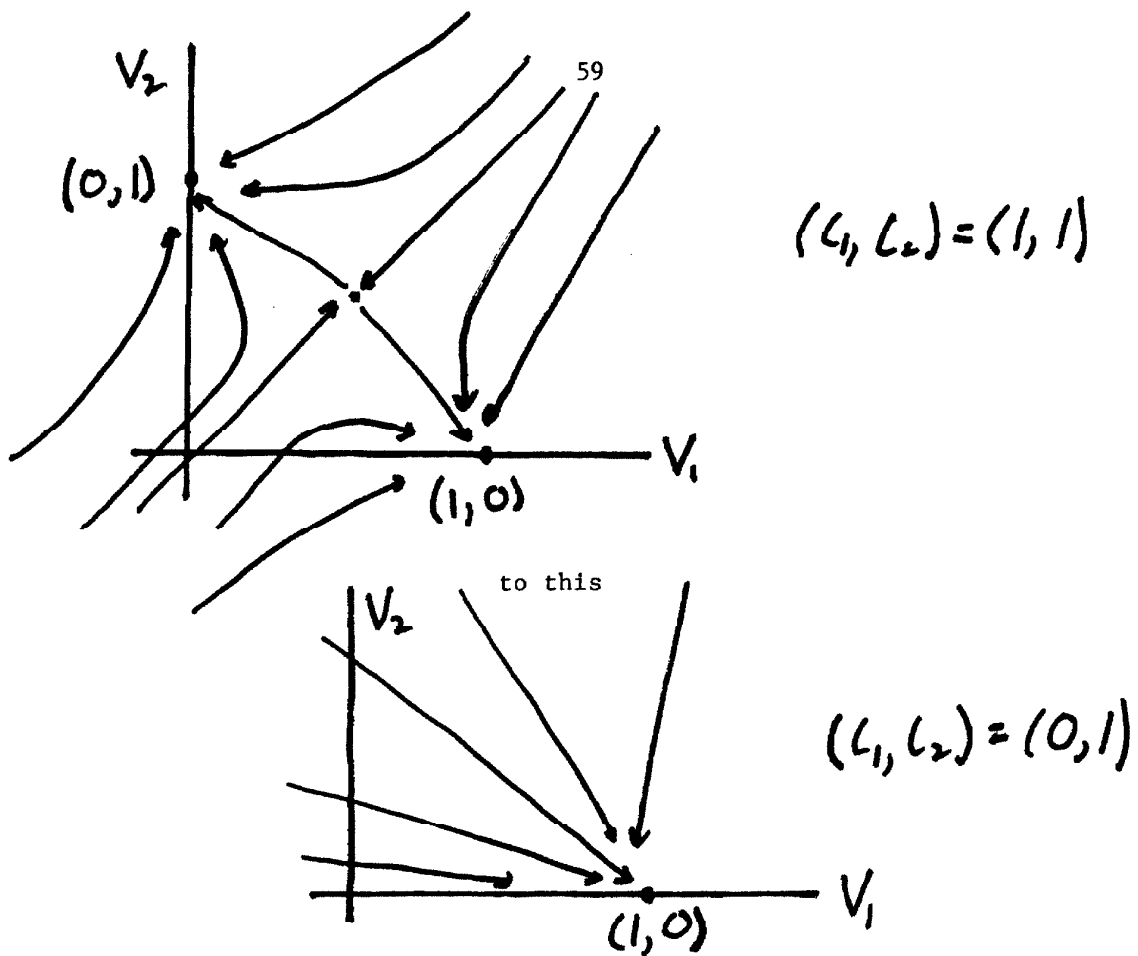
ANALYSIS OF SWITCHING CIRCUITS

1. INTRODUCTION

In this chapter we return to a more detailed analysis of storage and switching behaviour than was given in Chapter 1. We begin by using the techniques presented in Chapter 2 of this thesis to analyze in some detail the flip-flop; this time around, however, we consider the case of an R-S latch



with inputs. First we analyze the behavior of the latch in storage mode---when $i_1 = i_2 = 1$ (voltage corresponding to logical 1). Then we use the mathematics to analyze the switching behaviour of this device. That is, how the system changes when i_1 or i_2 is brought to 0. It will be seen that this corresponds to a bifurcation of the dynamical system in which its phase portrait changes from this



(There is a similar picture for the case $(i_1, i_2) = (1, 0)$). This bifurcation must involve the metastable equilibrium cancelling with one of the stable equilibria in such a way that the other stable equilibrium point is unaffected. During this bifurcation, every trajectory of the system is converted to one which enters the point $((1, 0)$ or $(0, 1))$ to which we are driving the system. This, in fact, is what the phrase "setting the latch" means in this context. When the input values are readjusted so that the device is in "storage mode" ($i_1 = i_2 = 1$), the system bifurcates back to the original phase portrait, but in such a way that any point sufficiently near to the driven state remains on a trajectory approaching that point. In this way the system can switch to storage mode ($i_1 = 1, i_2 = 1$) without affecting the stored value, provided enough time has elapsed in the driving or "set" condition

(i.e., $(i_1, i_2) = (1, 0)$ or $(0, 1)$), to insure that the voltages V_1 and V_2 are sufficiently close to the driven value. If the pulse $i_1 \downarrow \uparrow$ or $i_2 \downarrow \uparrow$ is either not high enough or not long enough, then the system will either not switch at all or will "hang"—i.e., will take an arbitrarily long time to settle into any recognizable storage state. This, I believe, is the essence of the synchronizer and arbiter "glitch" problem which has been known to occur in the design of asynchronous systems.⁽¹³⁾ There is no guarantee, in a system with completely asynchronous inputs, that the pulses will be long enough to cause the system to switch and settle properly. It seems the only solution to this problem is either a clock (thus making the system synchronous, but perhaps slower) or some kind of feedback link which communicates to the sender of the i_1 pulse that the latch has in fact latched.

With these preliminaries aside, we now turn to a more rigorous discussion of the metastable and switching behaviour of various devices. This discussion will draw heavily on the concepts presented in Chapter 2 of this thesis.

2. ANALYSIS OF THE R-S LATCH

In this section we will give a fairly detailed picture of the behaviour of an R-S latch under very general assumptions on the nature of the device. This section is broken into several subsections. The first subsection lays out the general assumptions we make; in the second we present the actual analysis of the device.

2.A Hypothesis on the Device

In this section we present the hypothesis we will make as to the exact nature of the R-S latch we will analyze in Section 2.B. These hypotheses will be on the nature of the driving dynamics of the latch and the way they change as we vary the inputs. It should be understood that by "latch" here we mean any device that exhibits certain kinds of dynamical behaviour; this behaviour will be outlined below. Thus the latch may be thermal, mechanical, magnetic, etc. In what follows we will refer to electrical devices for the primary examples; however, the generality of application should be understood.

We assume, then, that we are dealing with a device whose dynamical state is given by two voltages (or other physical quantities), V_1 and V_2 , which lie in an open set U about the set $[0, V_0] \times [0, V_0]$ (we assume $V_0 > 0$). We assume that the dynamics of the system in this region have the following properties:

(1) The system is given by a vector field on U which depends on two inputs i_1 and i_2 which are assumed to lie in some range which we take for convenience, to be $(-\varepsilon, V_0 + \varepsilon) = J$ where ε is a small positive parameter. That is, $(\dot{V}_1, \dot{V}_2) = (dV_1/dt, dV_2/dt) = (f_1, f_2)$, where f_1 and f_2 are infinitely differentiable functions of V_1, V_2, i_1 and i_2 .

(2) The vector field (f_1, f_2) is bounded on U ; that is, $f_1^2 + f_2^2 < M < \infty$ on U for all values of i_1 and i_2 .

(3) U has a boundary B which is a circle in the V_1, V_2 plane surrounding the square $[0, V_0] \times [0, V_0]$. We assume this circle is differentiable as a manifold. (This is a technical assumption which is easily verified in practice.)

(4) The vector field $\vec{V} = (\dot{V}_1, \dot{V}_2)$ extends to B in a smooth way. Then (2) implies $|\vec{V}|^2 = f_1^2 + f_2^2 \leq M$ on B , so \vec{V} is bounded.

(5) On B , \vec{V} points to the interior of U for all values of i_1 and i_2 ; also, $\vec{V} \neq 0$ on B for any choice of i_1, i_2 .

(6) \vec{V} has two attracting equilibrium points for (i_1, i_2) in some open S about (V_0, V_0) that is, there are two points in U , p_1 and p_2 , which we take in balls $B(p_1)$ and $B(p_2)$ about $(V_0, 0)$ and $(0, V_0)$, such that $\vec{V}(p_1) = \vec{V}(p_2) = 0$ and that the derivative of \vec{V} at these points has eigenvalues with negative real parts. The assumption on the eigenvalues of the derivative of \vec{V} is actually rather stronger than we will actually need, but is somewhat simplifying and so is held in this introductory treatment. At a point where $\vec{V} = 0$ its derivative is the matrix

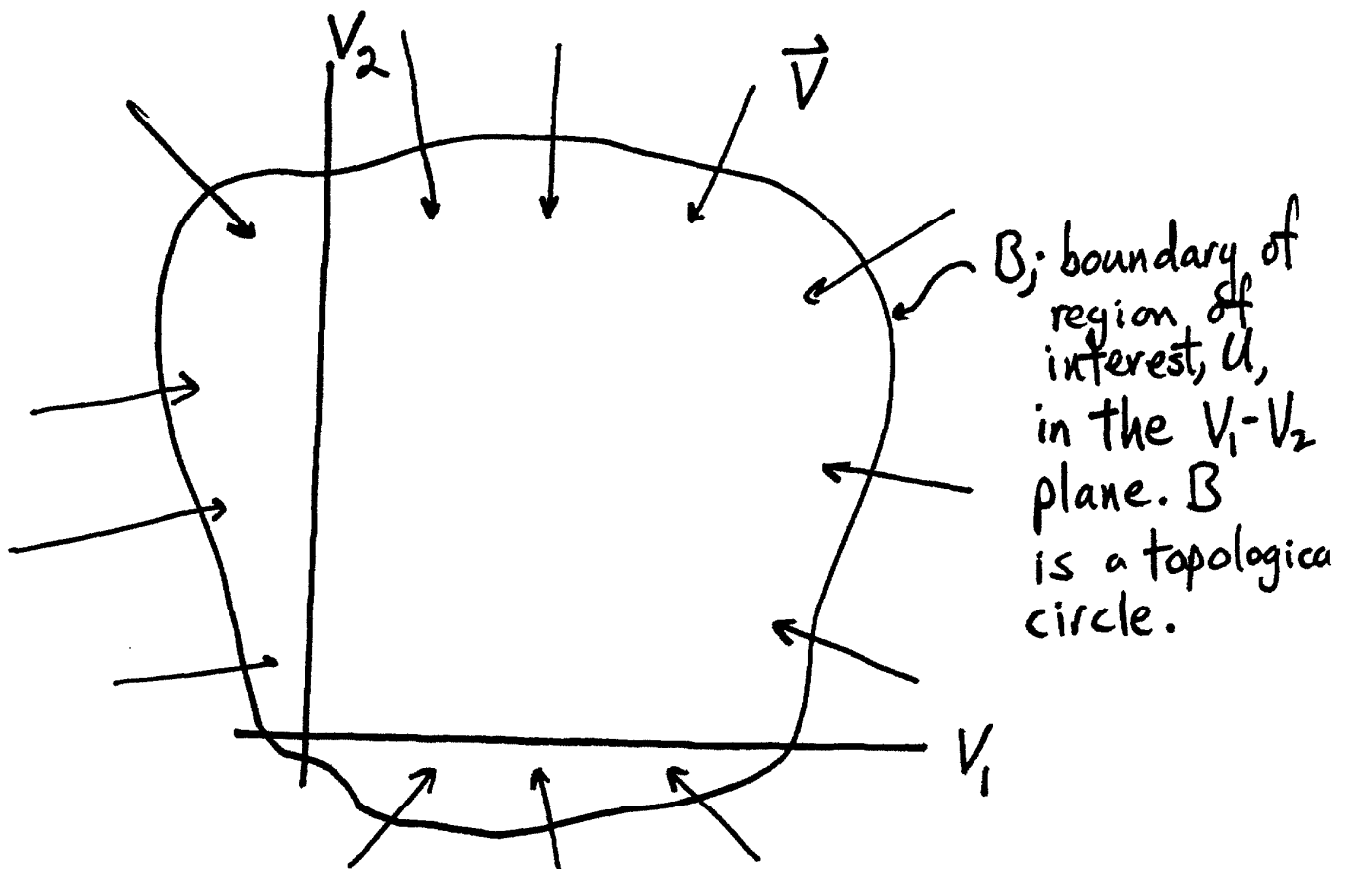
$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = D\vec{V} \quad .$$

The assumption on the eigenvalues of \vec{V} at its zeroes implies that about p_1 and p_2 there are open balls $B(p_1)$ and $B(p_2)$ such that if $q \in B(p_1)$, then $\phi_t(q, i_1, i_2) \rightarrow p_1$ as $t \rightarrow \infty$, where $\phi_t(i_1, i_2)$ is the flow of $\vec{V}(i_1, i_2)$; of course this implies $B(p_1) \cap B(p_2) = \emptyset$. For (i_1, i_2) in a sufficiently small neighborhood I_1 of $(0, V_0)$, \vec{V} has in the neighborhood $B(p_1)$ of p_1 , one zero, again having eigenvalues with negative real parts. For (i_1, i_2) in a sufficiently small neighborhood I_2 of $(V_0, 0)$ \vec{V} has a similar phase portrait, but the zero is now in the neighborhood of $B(p_2)$.

(7) As (i_1, i_2) moves from the small neighborhood I_1 of $(0, V_0)$ mentioned in Section 6 above to S , i.e., as i_1 moves from a neighborhood of 0 to a neighborhood of 1, if the system is in $B(p_1)$, then it remains in $B(p_1)$ for all values of i_1 as long as i_2 is close enough to V_0 . More precisely, there is a neighborhood T_1 of V_0 and a neighborhood T_0 of 0, $T_0 \cap T_1 = \emptyset$, and $I_1 = T_0 \times T_1$, $I_2 = T_1 \times T_0$ and $S = T_1 \times T_1$ such that if $q \in B(p_1)$, and $i_2 \in T_1$, then for all $i_1 \in J$, $\phi_t(q, i_1, i_2) \in B(p_1)$ for all $t \geq 0$. The corresponding statement with the roles of i_1, i_2 and p_1, p_2 reversed also holds.

This concludes the list of assumptions we make. We will now discuss some of the features of these hypotheses.

Assumption (5) gives us the following picture, independent of i_1 and i_2 :



What goes on inside the circle is still a matter of speculation, dear reader! But assumption (5) can be thought of as the mathematical formulation of the physical hypothesis that there is a power source driving (V_1, V_2) into a certain range, no matter what the inputs are.

Assumption (6) gives us at least two stable equilibria; the condition on the eigenvalues of $\vec{D}\vec{V}$ corresponds to a particularly strong form of stable point; but it is also the most likely to occur in a real device.⁽¹⁴⁾ The reader might also query the significance of open sets T_0, T_1 and others used in formulating these assumptions. The next figure illustrates these concepts:

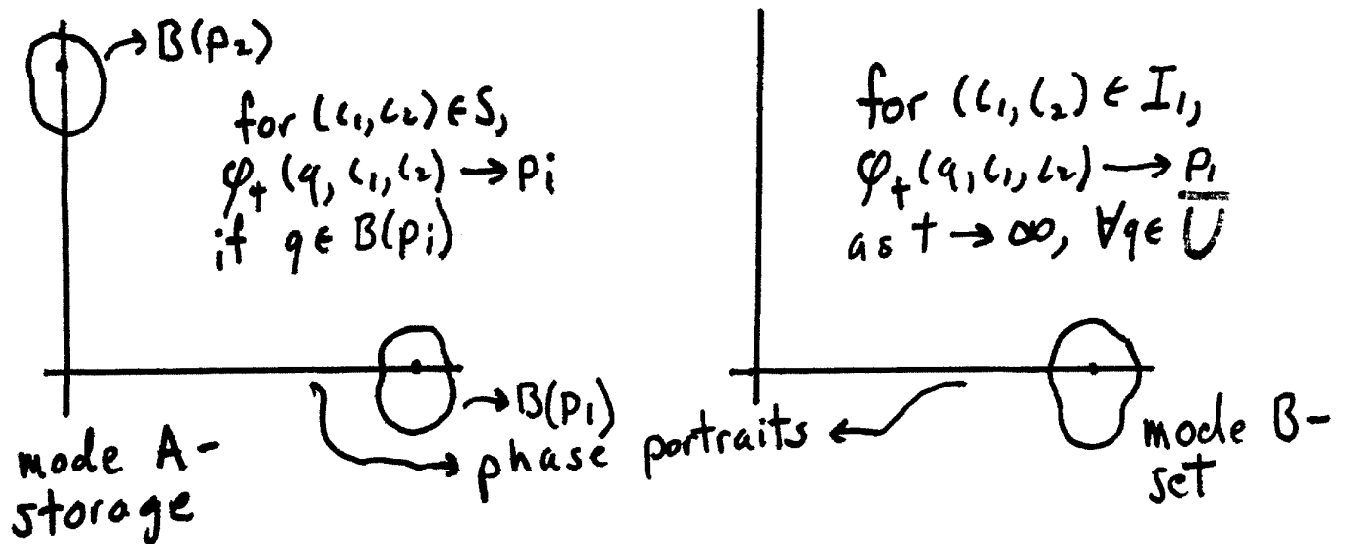
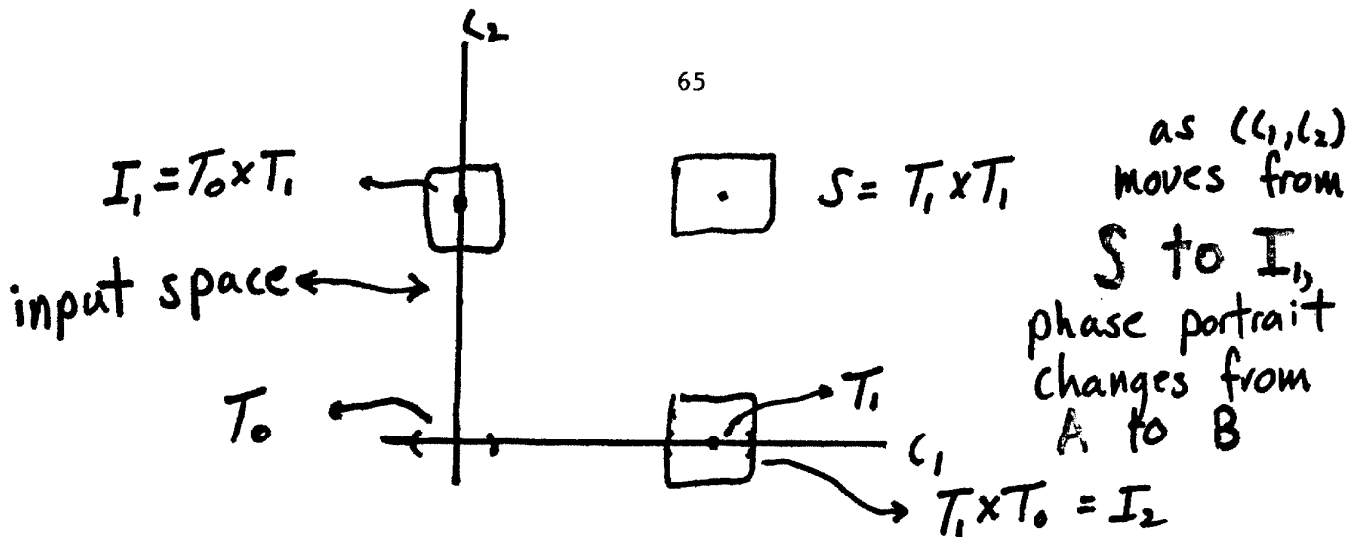


figure 2

The open sets $B(p_1)$, $B(p_2)$ are merely a way of saying that, though the exact location of the zeroes of \vec{V} can (in fact, will) vary with i_1 and i_2 , they are in certain well-defined ranges as long as i_1, i_2 are in the corresponding range.

Assumption (7) says that if the system is in a state sufficiently close to $(1,0)$ or $(0,1)$, i.e., has settled to an "exact" value, then changing an input will not have a large effect on that value if (1) we hold the other input high and (2) the system is in the proper internal

state. This is, in effect, the essence of programmable memory; it only resets itself if its input is opposite to its internal state.

This concludes the explication of the assumptions used in this analysis. The next section presents an analysis of the R-S latch, or "generalized latch" (as we have presented it), itself.

2B. We now proceed to investigate the consequences of our assumptions on the R-S latch. The first phenomenon we wish to discuss is the necessity of metastable operation for any device satisfying assumptions (1)-(7) above.

The term metastable operation refers to the possibility of a latch "hanging", i.e., taking an arbitrarily long time to settle recognizably in to a stable state. We will now show that this is, in fact, an essential feature of any bistable device. More precisely, let $B_i(r)$ denote the set

$$\{(v_1, v_2) | d((v_1, v_2) - p_i) \leq r\} \quad ,$$

where d is the ordinary distance function in the plane. Suppose r is such that $q \in B_i(r)$ implies $\phi_t(q) \rightarrow p_i$ as $t \rightarrow \infty$. (There always exists such an $r > 0$ for $(i_1, i_2) \in S$ by assumption (6); furthermore, if r_0 has this property, then any $r' < r_0$ has it as well.) The set $B_i(r)$ can be thought of as a "region of recognition"; once we are sure the trajectory of the system is in $B_i(r)$, we know the system is heading toward p_1 or p_2 , the stable states. Clearly, $B_1(r) \cap B_2(r) = \emptyset$. Now the definition of metastable behavior can be stated as follows: for any r such that $\phi_t(B_i(r)) \rightarrow p_i$ as $t \rightarrow \infty$ and all $T > 0$ no matter how large, there is an

open set H (for "hang") of values (V_1, V_2) such that for all $q \in H$, there is no $t_0 \leq T$ such that $\phi_{t_0}(q) \in B_r(p_i)$. The fact that H is an open set means it "takes up space" in a topological sense; it also implies that H has positive area; thus if we assume a uniform distribution of initial values, there is a finite probability the circuit will not settle within time T for all $T > 0$. Assuming a distribution ρ of initial values, if $\rho > 0$ on the set of values under consideration (the set bounded by the circle in assumption (5)), $\int_H \rho dV_1 dV_2 > 0$ so for many realistic distributions of initial values, there is a finite nonzero probability the circuit will not have settled within time T .

The first step in proving these assertions is to show that assumptions (1)-(7) imply the existence of at least one other equilibrium point in the system under consideration. To do this we need to develop a theorem which relates the number of zeroes of a vector field on a disk to global topological properties of the disk. This theorem tells us how the indices (cf. Chapter 2 of this thesis) of a vector field relate to the topology of the underlying manifold. The proposition, it will be seen, takes a different form for even or odd dimensional disks; we will prove both forms here since they will both be useful in this discussion.

Theorem: Let D^n be the closed n ball, i.e.,

$$D^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid \sum (x^i)^2 \leq 1\},$$

and let \vec{V} be a vector field on D which points inward on ∂D where

$$\partial D^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid \sum (x^i)^2 = 1\}.$$

Suppose \vec{V} has only isolated zeroes on D . Then if n is even

$$\sum_{\substack{p_i \text{ a zero} \\ \text{point of } \vec{V}}} \Theta_{\vec{V}}(p_i) = 1 ,$$

if n is odd

$$\sum_{\substack{p_i \text{ a zero} \\ \text{point of } \vec{V}}} \Theta_{\vec{V}}(p_i) = -1 ;$$

or, in general

$$\sum_{\substack{p_i \text{ a zero} \\ \text{point of } \vec{V}}} \Theta_{\vec{V}}(p_i) = (-1)^n . \quad (X)$$

Proof: This is proved using the following fact relating the Euler-Poincare characteristic of two subspaces X_1, X_2 of a space X to $\chi(X_1 \cup X_2)$ and $\chi(X_1 \cap X_2)$.

Fact: Let X_1, X_2 be compact submanifolds of a compact manifold X .

Then

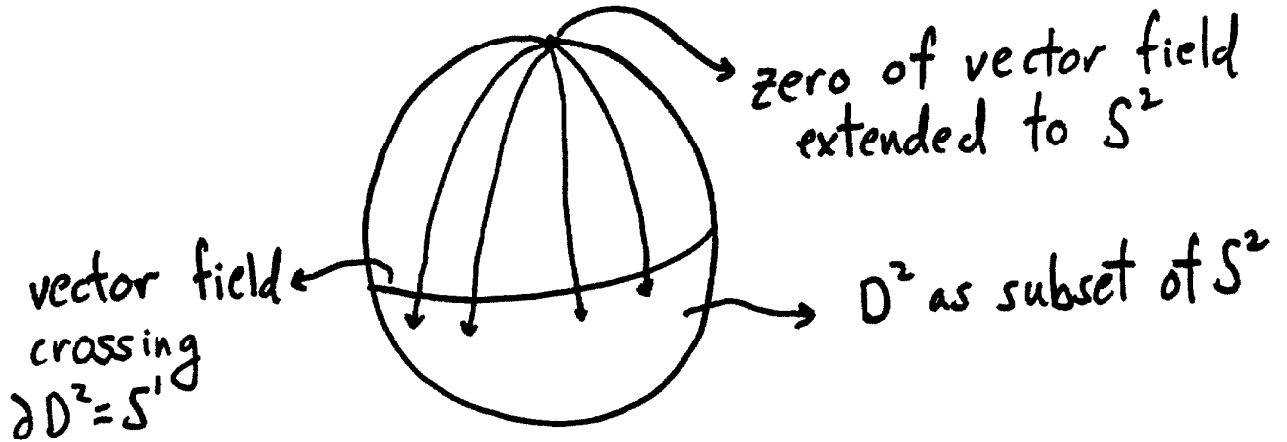
$$\chi(X_1 \cup X_2) + \chi(X_1 \cap X_2) = \chi(X_1) + \chi(X_2) . \quad (A)$$

For an exposition, cf. (15).

We apply this fact to the theorem we are trying to prove as follows: We can consider the disk D^n as part of an n -sphere S^n ; since \vec{V} points in on ∂D^n , we can extend \vec{V} to the entire sphere in such a way

that it has exactly one zero of index one in the complement of D^n in S^n ;

the next figure illustrates the two-dimensional case:



To be precise, on another copy of D^n define a vector field \vec{V}_1 as follows: on ∂D^n , $\vec{V}_1 = -\vec{V}|_{\partial D^n}$. Define $\vec{V}_1(0) = 0$. In the remainder of D^n , set $\vec{V}_1(q) = \|q\| \vec{V}_1(q/\|q\|)$, i.e., at q we find \vec{V}_1 by drawing a line segment between 0 and $q \neq 0$, extending it to meet $\partial D^n =$ an $n-1$ sphere in \mathbb{R}^n , and choosing $\vec{V}_1(q)$ to be the value of \vec{V}_1 at the point where this line intersects ∂D^n (on the side of q), and scaling the magnitude of \vec{V}_1 by $\|q\|$, where $\|q\|$ is the Euclidean distance

$$\|q\| = \sqrt{\sum_{i=1}^n (x_i(q))^2}$$

This field \vec{V}_1 on D^n has index 1 (it is homotopic to the identity field $\vec{V}_1 = (x_1, \dots, x_n)$). Furthermore, if we construct S^n out of two copies of D^n glued together along ∂D^n , \vec{V}_1 and \vec{V} meet to form a smooth field \vec{V} on S^n . We can now apply formula (A) of p. 68. By the Poincare-Hopf theorem (cf. Chapter two of this thesis), taking X_1 and X_2 to be the two copies of D^n ,

$$\sum_{p_i \in S^n} \Theta_{\vec{V}}(p_i) = \chi(X_1 \cup X_2) .$$

Since the 0 of \vec{V}_1 has index 1, and the other zeroes of our field \vec{V} on S^n are in the original copy of D^n , we have

$$1 + \sum_{p_i \in D^n} \Theta_{\vec{V}}(p_i) = \chi(X_1 \cup X_2),$$

so

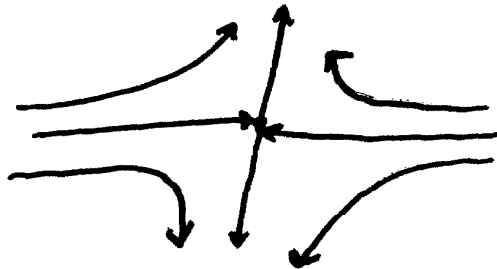
$$1 + \sum_{p_i \in D^n} \Theta_{\vec{V}}(p_i) + \chi(X_1 \cap X_2) = \chi(X_1) + \chi(X_2) = 2\chi(X_1) = 2\chi(D^n) = 2$$

since $X_1 = X_2 = D^n$, and $\chi(D^n) = 1$ since it is contractable. Thus

$$\sum_{p_i \in D^n} \Theta_{\vec{V}}(p_i) + \chi(X_1 \cap X_2) = 1 .$$

Now $X_1 \cap X_2$ is, in the case at hand, an $n-1$ sphere, so $\chi(X_1 \cap X_2) = 0$ if n is even and 2 if n is odd (cf. (16)). The theorem follows from this.

We can use this result to demonstrate the existence of the third equilibrium point in the vector field at hand. Since each of the zeroes postulated in assumption (7) above has index one, formula X of p. 68 implies the existence of at least one zero with index -1. Now in two dimensions, a zero of index -1 is what is known as a saddle; that is, $D\vec{V}(p)$ has real eigenvalues of opposite signs. (It can be shown that if $\det(D\vec{V}(p)) \neq 0$ at a 0 of \vec{V} , the index of \vec{V} at p is just $\text{sign}(\det(D\vec{V}(p)))$.) Now, near a saddle \vec{V} must look like this



cf. (17). Note that a saddle point p has two incoming trajectories and two outgoing trajectories, and the rest of them head in and then out as $t \rightarrow \infty$.

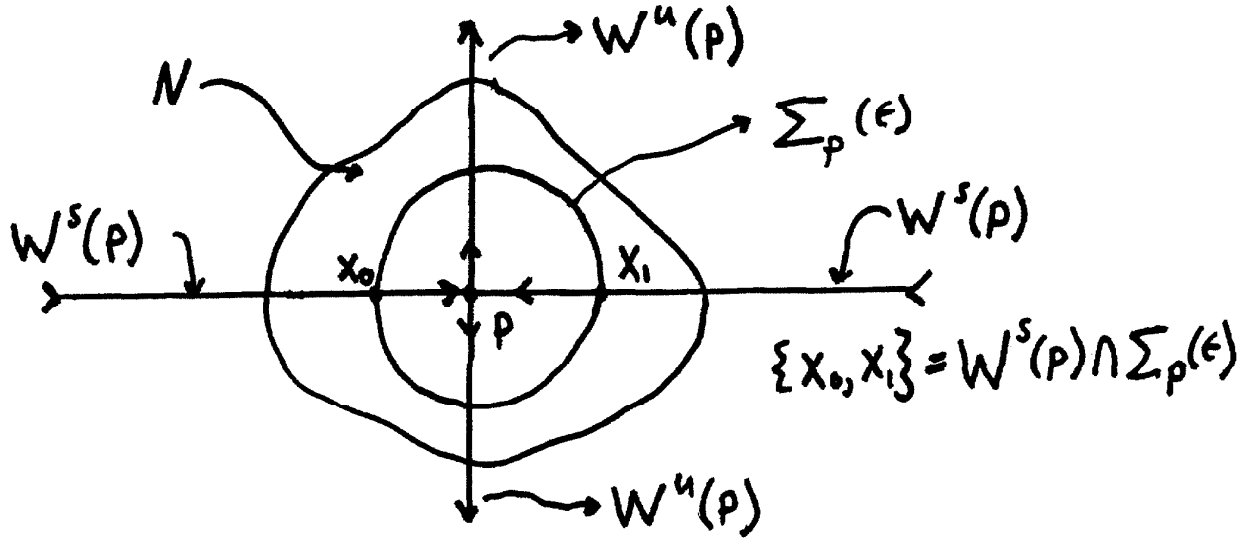
These special trajectories are called the stable and unstable manifolds of p , respectively. They are often denoted $W^S(p)$ and $W^U(p)$. That is,

$$W^S(p) = \{x \in \bar{U} \mid \phi_t(x) \rightarrow p \text{ as } t \rightarrow \infty\}$$

$$W^U(p) = \{x \in \bar{U} \mid \phi_t(x) \rightarrow p \text{ as } t \rightarrow -\infty\}$$

Actually, we have only proven the existence of at least one saddle, i.e., there could be more if we allow more zeroes of \vec{V} of index 1. Nevertheless, this can only complicate the behaviour of the latch, because it will create more equilibrium points. For simplicity we will assume a situation where things are as simple as possible, i.e., we have two stable equilibria as postulated and exactly one saddle as is forced on us by our assumptions. Now the saddle equilibrium, which we will also refer to by the term "metastable equilibrium" or "metastable point", allows us to find the set H referred to above. We do this as follows: for any point q on the trajectories heading into the metastable point p as $t \rightarrow \infty$, $\phi_t(q)$ approaches p but never reaches it. Pick any $T > 0$, any $r > 0$ such that $\phi_t(B_1(r)) \rightarrow p_1$ as $t \rightarrow \infty$. Next we note that there is a neighborhood of p , the metastable point, which is disjoint from $B_1(r) \cup B_2(r)$. This is because $B_1(r) \cup B_2(r)$ is a closed subset of the space $\bar{U} = U \cup B$ and $p \notin (B_1(r) \cup B_2(r))$. For if $q \in B_1(r)$, then $\phi_t(q) \rightarrow p_1$ as $t \rightarrow \infty$; but $\vec{V}(p) = 0$ so $\phi_t(p) = p$ for all t . Thus $p \notin B_1(r) \cup$

$B_2(r)$. Since \bar{U} is compact and p is not on B by assumption (5), the fact that $\bar{U} = U \cup B$ is a normal space (see (7)) implies there is an open neighborhood N of p with $N \cap B_1(r) = \emptyset$. Within this neighborhood we can choose a small sphere $\Sigma_p(\epsilon)$. If we choose $\epsilon > 0$ small enough, this sphere intersects the stable and unstable manifolds of p each exactly twice--once on each local component, as pictured below:

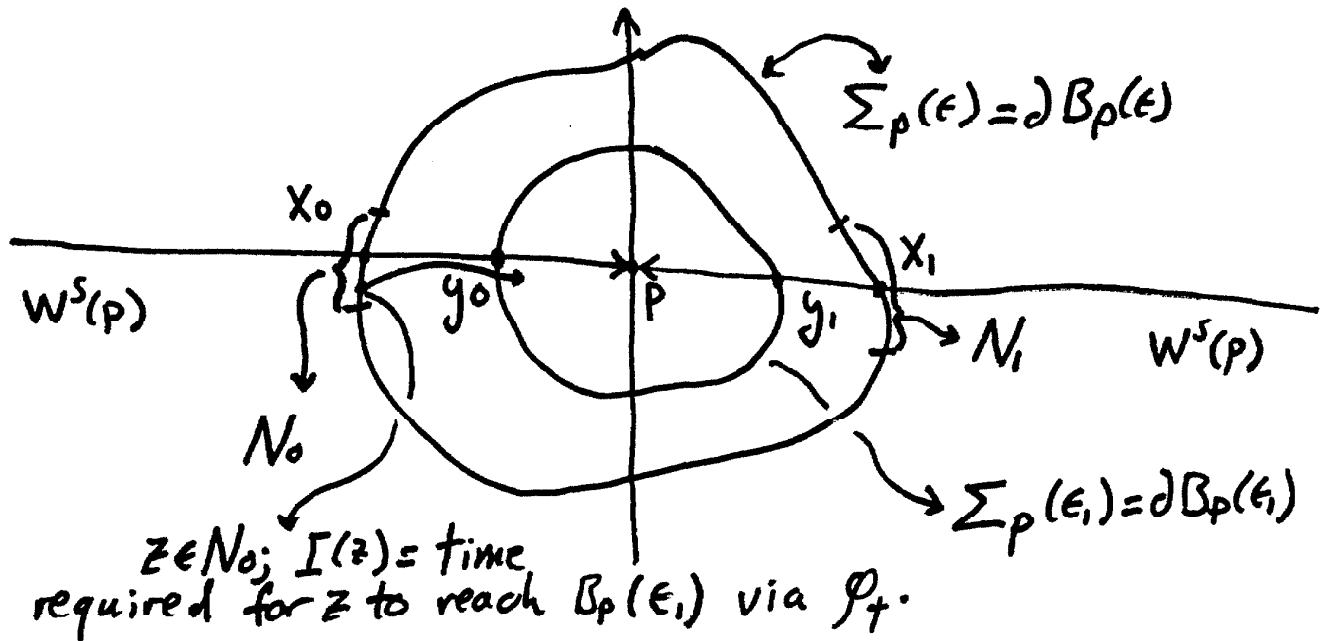


Let the points in $W^s(p) \cap \Sigma_p(\epsilon)$ be x_0 and x_1 . Then there is no finite $t > 0$ such that $\phi_t(x_0) = p$ or $\phi_t(x_1) = p$, but $\lim_{t \rightarrow \infty} \phi_t(x_0) = \lim_{t \rightarrow \infty} \phi_t(x_1) = p$ as $t \rightarrow \infty$.

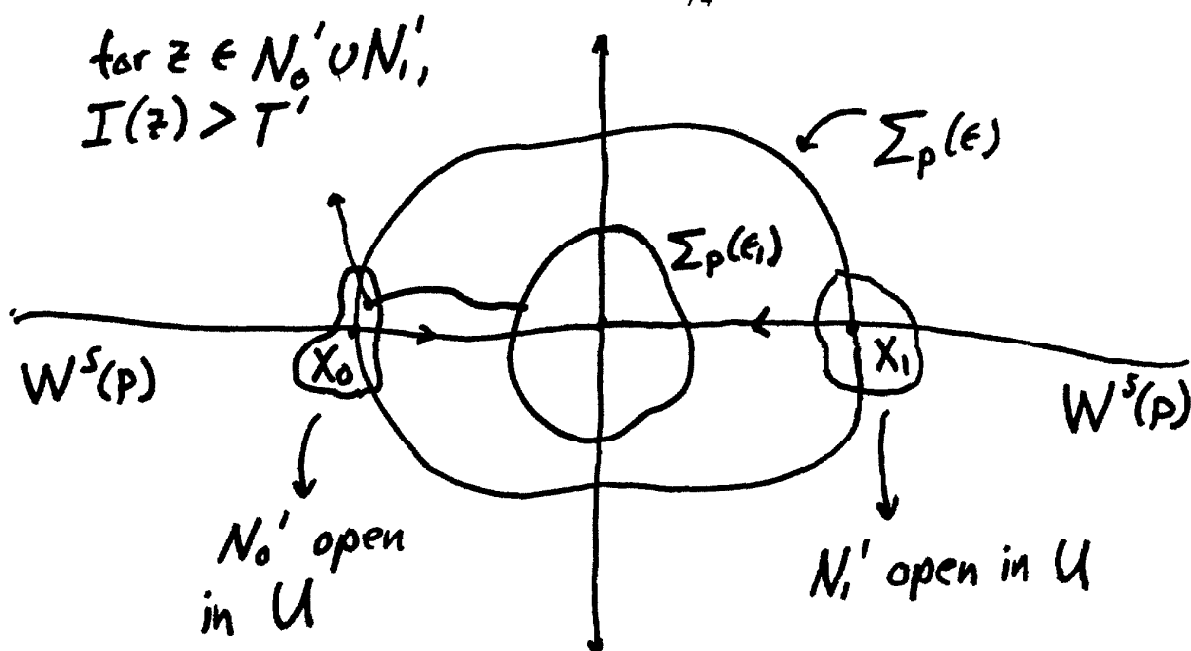
Pick $T > T$. Consider the points $\phi_{T^{-1}}(x_0)$ and $\phi_{T^{-1}}(x_1)$. Since neither of these points is p , there is another ball $B_p(\epsilon_1)$ such that $\phi_{T^{-1}}(x_i) \in B_p(\epsilon) - B_p(\epsilon_1)$ for $i = 0$ or 1 . Let y_0, y_1 be the points where $W^s(p)$ intersects $B_p(\epsilon_1)$, y_i in the orbit of x_i . Pick a small neighborhood N_1 of x_1 in $\Sigma_p(\epsilon)$. For $z \in N_1$, define $I(z)$ to be the time required for z to reach $B_p(\epsilon_1)$. That is,

$$I(z) = \inf\{t \mid \phi_t(z) \cap B_p(\epsilon_1) \neq \emptyset\}.$$

The picture below illustrates these ideas



Note that since at x_1 \vec{V} points inward, we can choose N_1 small enough so that \vec{V} points inward on N_1 . So $I(z) > 0$ for $z \in N_1$. Also, choosing N_1 sufficiently small insures $\phi_t(z)$ hits $B_{\epsilon_1}(p)$ for some $t < \infty$, if $z \in N_0 \cup N_1$. Thus we can take $I(z) < \infty$. The critical thing about I is that, on $N_0 \cup N_1$, it is differentiable. Now $I(x_1) > T' > T$, since $\phi_t(x_1)$ hits y_1 on $B_{\epsilon_1}(p)$. Differentiability of I implies it is continuous, so we can, by taking N_1 small enough, insure $I(z) > T'$ for $z \in N_0 \cup N_1$. Furthermore, we can thicken the N_1 slightly to open sets N'_1 in U such that, extending I to $N'_0 \cup N'_1$, $I(z) > T'$ in $N'_0 \cup N'_1$. See the next figure



We have now constructed an open set, namely $H' = N_0' \cup N_1'$, with the property that for each $z \in H'$, $\phi_{T'}(z)$ is not in $B_p(\epsilon_1)$, but for some $t > T'$, $\phi_t(z) \in B_p(\epsilon_1)$. This means that by time $T' > T$ we will, if we start in H' , not even have reached $B_p(\epsilon_1)$, hence could not possibly have reached one of the $B_r(p_i)$, our "regions of recognition." We can extend this set H' as follows. Set

$$H = \bigcup_{t \geq 0} \phi_{-t}(H'),$$

i.e., H is the set of all points which eventually move into H' under the action of the flow of \vec{V} . For $z \in H$, the time required to enter $B_p(\epsilon_1)$ is finite, since there is $t \geq 0$ with $\phi_t(z) \in H'$ and any point in H' reaches $B_p(\epsilon_1)$ in finite time. However, since any point in H' takes more than $T_1' > T$ to reach $B_p(\epsilon_1)$, any point in H takes more than $T_1' > T$. Furthermore, H' is open, since H is open and each ϕ_t is a diffeomorphism. Thus we have established the existence of the claimed set H for any $T > 0$ and r with $\phi_t(B_{p_i}(r)) \rightarrow p_i$ as $t \rightarrow \infty$.

Note that the set H we have constructed is not maximal by any means - there may be a larger set satisfying our requirements. But H does have positive area, as claimed, being an open set. Thus under any distribution of initial values ρ on \bar{U} with $p > 0$ (a realistic assumption since there is a small probability of the system being initialized to any point), there is a positive probability the system will not be in a recognizable state within time T .

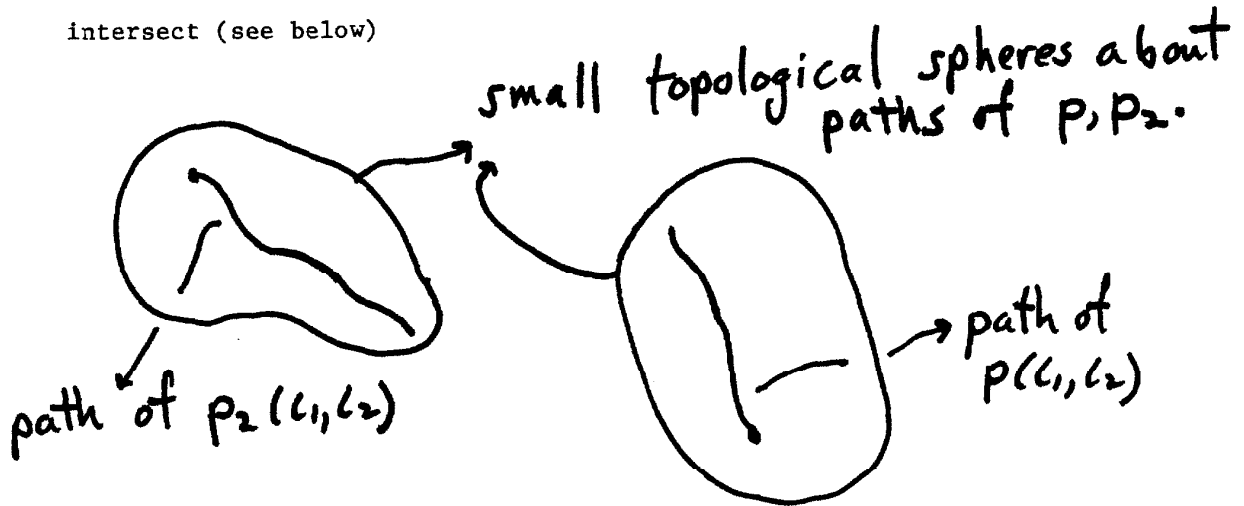
We now wish to show how the topological method of analysis used here can be exploited to understand the switching behaviour of the latch as we change i_1 or i_2 . For convenience we will assume i_2 is held in the neighborhood T_1 of V_0 and i_1 varies from T_1 to T_0 ; the case in which i_2 varies while i_1 is held fixed we assume to be symmetrical.

The first thing we need to show is that the position of the zeroes of \vec{V} move continuously with i_1, i_2 as they vary. We will do this under the previously stated assumption that no new zeroes of \vec{V} are created, since this represents the simplest (and probably the most well designed) kind of latch. What we wish to show, then, is that for every $\epsilon > 0$ and i_{10}, i_{20} there is a $\delta > 0$ such that if $|i_1 - i_{10}| + |i_2 - i_{20}| < \delta$, then for any zero p_i of $\vec{V}(i_1, i_2)$ of nonzero index, $\|p_i(i_1, i_2) - p_i(i_{10}, i_{20})\| < \epsilon$.

So pick $\epsilon > 0$. About any zero $p_i(i_{10}, i_{20})$ of $\vec{V}(i_{10}, i_{20})$ construct a small sphere $\sum_{\epsilon_1}(p_i)$ of radius $\epsilon_1 < \epsilon$ with no zero other than p_i in or on this sphere. Now we know (cf. chapter 2 of this thesis) that for small changes in \vec{V} , the index $\theta_{\vec{V}}(p_i)$ about p_i does not change. Since the index of each zero of \vec{V} is ± 1 , and a field with no zero in $\sum_{\epsilon_1}(p_i)$ has index 0, the zero of \vec{V} in $\sum_{\epsilon_1}(p_i)$ must remain in $\sum_{\epsilon_1}(p_i)$ for sufficiently small

changes in \vec{V} ; since \vec{V} varies smoothly with i_1, i_2 , there must be a δ such that for $|i_1 - i_{10}| + |i_2 - i_{20}| < \delta$, the zero of \vec{V} remains in $\sum_{\epsilon_1}(p_1)$; this shows the positions of the zeroes depend continuously on (i_1, i_2) . We will denote this position by $p_i(i_1, i_2)$.

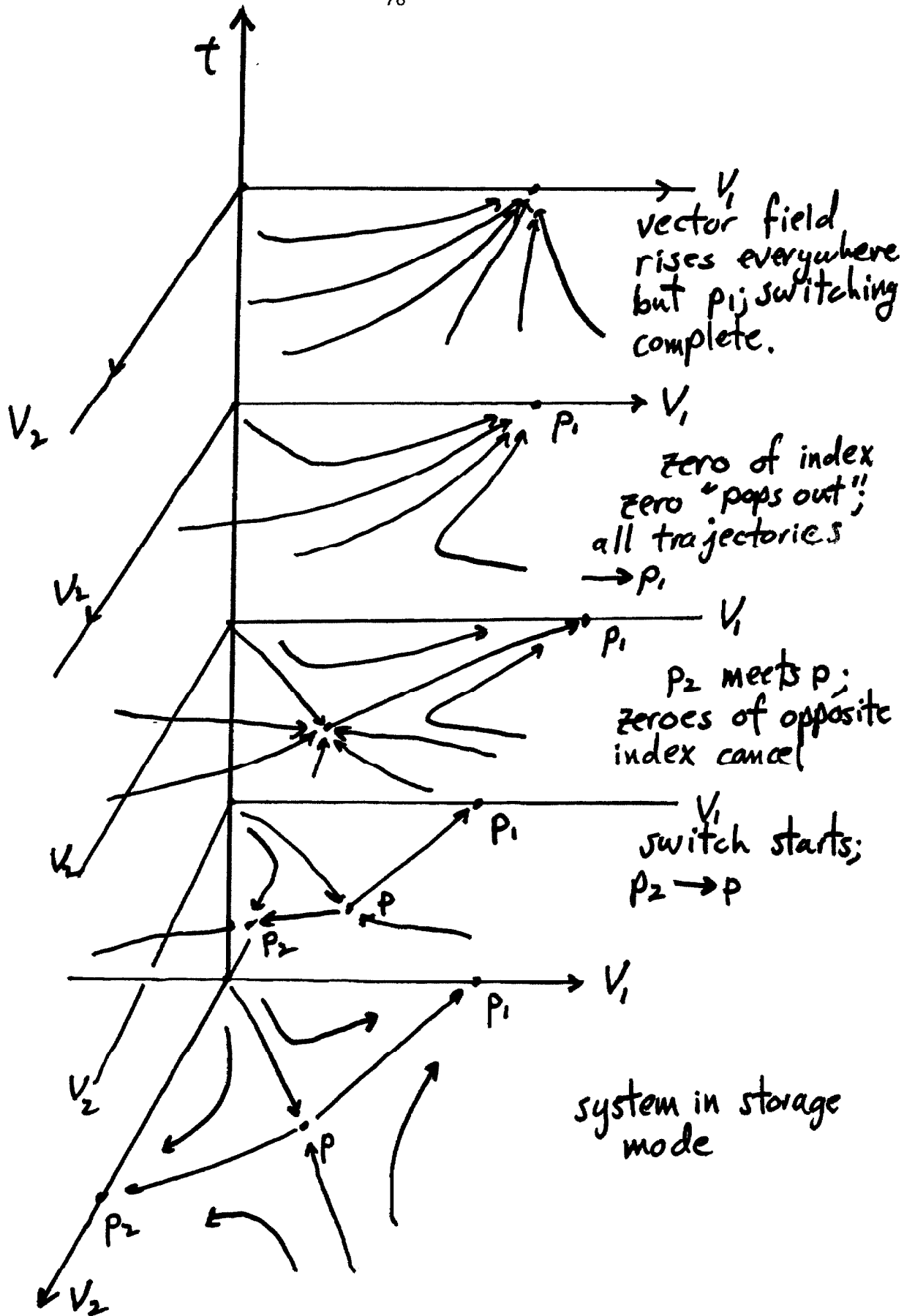
Now we can use the continuity of $p_1(i_1, i_2)$ (as long as $\Theta_v(p_1) \neq 0$) to show that, as i_1 moves from T_1 to T_0 , there must be a value of i_1 where the metastable point actually meets the zero p_2 which is near $(0, V_0)$ for $(i_1, i_2) \in S$. Specifically, let i_1, i_2 now depend on a parameter s which varies in $[0, 1] = J_1$. We assume $i_2(s) \in T_1$ for all $s \in J_1$, and $i_1(0) \in T_1$, $i_1(1) \in T_0$. (One would think of s as a time parameter on which the inputs depend.) Furthermore we assume $i_1(s_1), i_2(s)$ are smooth functions. Then $p_1(i_1, i_2)$ is a continuous map of J_1 into U . Suppose p_2 and p do not meet. Then, since the path of each zero is compact, the two paths will remain some finite distance ϵ apart. Then we could construct a small topological sphere about each path, and these spheres will not intersect (see below)



We compute the index of \vec{V} using these spheres. Since no zero of \vec{V} crosses either sphere, the index of \vec{V} at p does not change as we vary s

from 0 to 1. When $s = 1$, we have $(i_1(s), i_2(s)) \in I_1$; when $s = 0$, $(i_1(s), i_2(s)) \in S$. Now the zero p_1 in the vicinity of $(V_0, 0)$ must stay inside by assumption (7); otherwise a trajectory would leave $B(p_1)$. Furthermore, neither other zero of \vec{V} can enter $B(p_1)$; otherwise, when we ran s from 0 to 1 (returning to storage mode) a trajectory would leave $B(p_1)$, contradicting one of our assumptions. Therefore, unless two zeroes meet, the phase portrait of \vec{V} cannot smoothly change as postulated; because the index of \vec{V} computed about any sphere with no zero inside is zero. If we compute the index of \vec{V} using one of the spheres about the path of p_2 or p we have just constructed, it will be zero for the phase portrait B in figure Z which we reach for $(i_1, i_2) \in T_0 \times T_1$ but we have just shown it to be ± 1 . This contradiction shows that the zero p must meet p_2 as i_1 moves from T_1 to T_0 in the fashion described above. Furthermore, this meeting cannot take place within $B(p_1)$ or else a trajectory heading toward p_1 as $t \rightarrow \infty$ would "break off" and head toward p or p_2 - or start to cycle around as i_1 moves from T_0 to T_1 (we assume such cycling to be explicitly forbidden).

Thus one could think of the switching behaviour of the latch as occurring precisely at the moment when the zeroes p_2 and p meet--when the dynamical system changes its qualitative behaviour from having two stable states to having one. We can make a "space-time" graph of this; the next figure accomplishes this.



Thinking in terms of this picture, one can understand the effect of runt pulses and pulses whose duration is too short to cause the system to stay in mode B of figure 2 above, i.e., to get into $B_1(p_1)$. If a pulse is not low enough, the zeroes will never meet and no switching will occur. If it is strong enough to cause switching, but just barely, the system will take a long time to settle because the vector field \vec{V} is still very near zero and hence the "velocity", or rate of traverse of V_1, V_2 with t , is small. If the pulse is strong enough to cause good switching, i.e., $|\vec{V}|$ rises so it is strongly nonzero where p_2, p came together, but is of too short a duration the system will not have time to get close enough to p_1 before the system switches back into storage mode. Hence the trajectory it is on may, when i_1 rises to 1, reconnect to p or p_2 and the system will have the wrong value even though it has received a pulse of the correct strength. The only solution to this problem is to make sure all pulses are high enough and long enough; the only way this seems possible in a completely asynchronous system is by feedback links which report the occurrence of transitions to the sender.

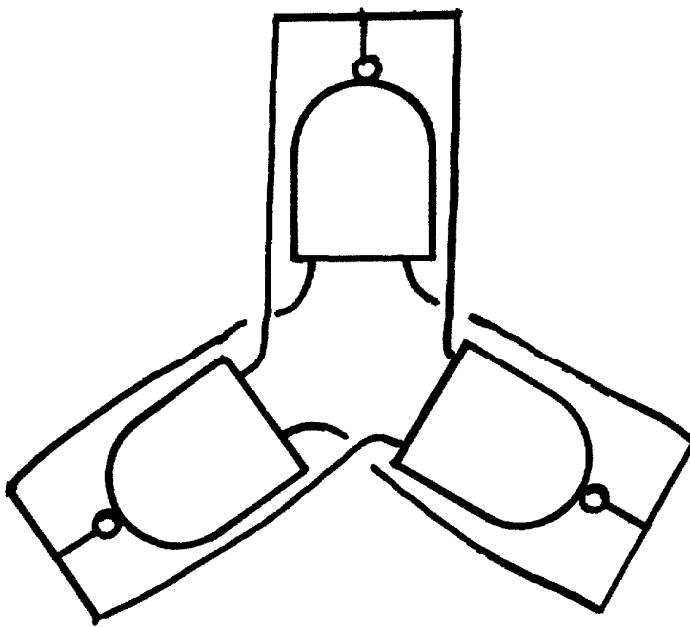
Obviously there is some trade-off between time and energy here; a weak pulse, just strong enough to cause switching, should, if held long enough, cause the system to store a new value. A stronger pulse should take less time. But there will always be a minimum time and pulse strength for a given device.

This concludes our analysis of the two-state latch. We now turn to a brief view of a three-state device.

3. THE TRI-FLOP

In this section we give a brief indication of how the techniques used here can give a picture of the metastable behaviour of a more complex circuit--the tri-flop (or flip-flap-flop in Knuth's terminology (18)), first brought to our attention by Seitz (19).

We consider a circuit whose schematic is given below



Elementary logic shows this circuit has three stable states--one with each of the nodes 1, 2, 3 at V_0 and the other two at 0. We envision the configuration space of this to be a three-dimensional version of that of the flip-flop. That is, we have three voltage axes V_1, V_2, V_3 with attracting equilibria at $p_1 = (V_0, 0, 0)$, $p_2 = (0, V_0, 0)$ and $p_3 = (0, 0, V_0)$. We imagine also a sphere surrounding these points on which the vector field $\vec{V} = (dV_1/dt, dV_2/dt, dV_3/dt)$ points into the region containing the points p_i . We will investigate the occurrence of other, metastable equilibria.

Now three is odd, so formula X of theorem one applies with $(-1)^n = -1$; so

$$\sum_{p_1 \text{ a zero } \vec{V}} \Theta_{\vec{V}}(p_1) = -1.$$

In all three dimensions, attractors have index -1 , so the three stable points p_1 contribute a total of -3 ; thus we have

$$-3 + \sum_{\text{metastable points}} \Theta_{\vec{V}}(p_i) = -1 ,$$

or

$$\sum_{i>4} \Theta_{\vec{V}}(p_i) = 2 ,$$

where p_i for $i > 4$ are the equilibria we are looking for. Next we assume that the system is completely symmetrical with respect to the group action interchanging the p_i . We do this for simplicity, but it seems likely that a device made from nearly identical components will exhibit close to this behavior. If the system is initialized at (V_1, V_1, V_1) it should move along the line $\ell = (1,1,1)s$ ($-\infty < s < \infty$) by symmetry. A reasonable assumption, based on the observed behaviour of such devices, is that at (V_0, V_0, V_0) all voltages decrease ($dV_i/dt < 0$) and at $(0,0,0)$ all voltages increase ($dV_i/dt > 0$). Then along the line ℓ , there is a zero p_4 of \vec{V} . This zero has one incoming direction, along ℓ . Symmetry demands, then, that the other two directions at p_4 , (eigenspaces of $D\vec{V}$ at p_4) either both enter or both exit if $D\vec{V}$ has eigenvalues with nonzero real parts. Thus the determinant of $D\vec{V}$ at p_4 , being the product of its eigenvalues, is negative. Thus $\Theta_{\vec{V}}(p_4) = -1$ since the index is equal to the sign of the determinant of $D\vec{V}$ at p_4 . If we assume there are no attractors other than p_1, p_2, p_3 , then p_4 must have a two-

dimensional unstable manifold. Thus p_4 is a metastable point. Now our index formula yields

$$-1 + \sum_{i>5} \theta_{p_i}(\vec{V}) = 2$$

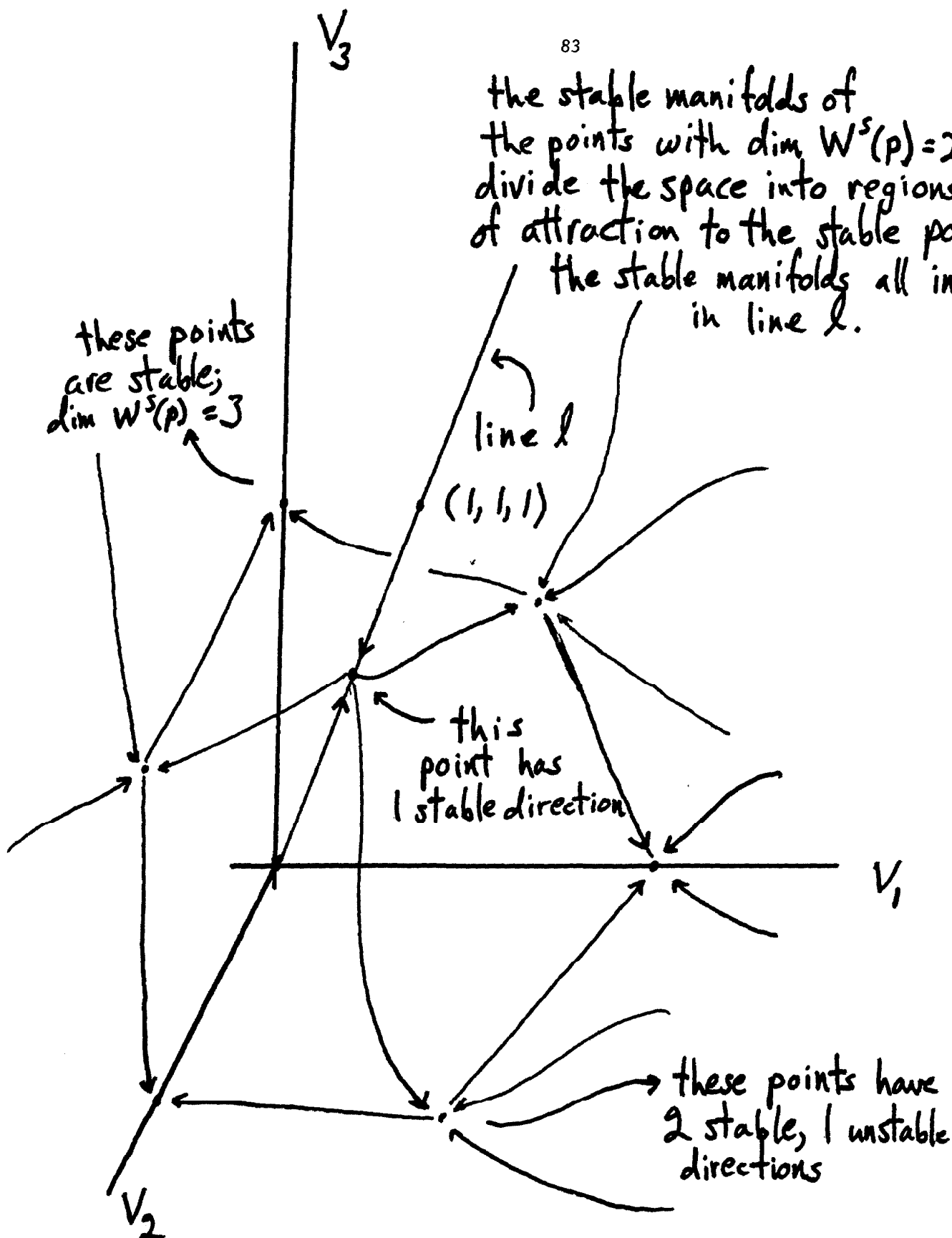
or

$$\sum_{p_i > 5} \theta_{p_i}(\vec{V}) = 3 \quad .$$

The simplest way we can meet this constraint is by having three zeroes of index +1. These will either be repellers or points with two directions of entry (corresponding to two negative real parts of eigenvalues of $D\vec{V}$) and one of exit (corresponding to a positive real part of an eigenvalue of $D\vec{V}$). If we assume there are no repellers (completely unstable equilibria) then we have points p_5, p_6, p_7 where $\vec{V} = 0$ with two directions of entry and one of exit. Symmetry implies they will be symmetrically placed about ℓ , or all on ℓ . We assume none are on ℓ . Then they are symmetrically placed about ℓ .

We now make an intelligent guess that the phase portrait of the tri-flop is as pictured below

the stable manifolds of the points with $\dim W^s(p) = 2$ divide the space into regions of attraction to the stable points; the stable manifolds all intersect in line ℓ .



We see that even a simple device made from three hand gates hooked up in the above fashion has at least four metastable points, and that the flow is partitioned neatly by the stable and unstable manifolds of these points into regions which are attracted to different attractors in the phase space. With more detailed, rigorous methods a general dynamical theory of switching may emerge which discusses quite general changes of state with input. The author is currently developing such a model.

4. EPILOGUE AND DIRECTIONS FOR FUTURE RESEARCH

In this paper we have tried to show how topological methods can be used to analyze the switching behaviour of some simple devices, and hopefully have showed a direction in which the general theory might develop. Though the methods used were not always completely rigorous we believe they and the conclusions are essentially correct. It seems like a general dynamical theory of switching could be developed to treat many of these problems in a systematic way. The author is currently engaged in this pursuit.

Notes - Switching Dynamics

1. Smooth in this context means dV_i/dt , $i \in \{A, B\}$, is a sufficiently differentiable function of V_A, V_B . See below, p. 5.
2. For any two sets Z and W , $Z \times W$ denotes the Cartesian product of Z and W , or $\{(z, w) | z \in Z, w \in W\}$.
3. A point p is said to be an isolated zero of a vector field V iff $V(p) = 0$ and there is a small open set U , $p \in U$, and $V|_{(U-p)} \neq 0$.
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11. Ibid.
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13. The Synchronizer "Glitch" Problem, Washington University Technical Report no. 47, by Thomas S. Chaney.
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15. Spanier, Chapter 4, Exercise B2. See note 10.
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17. Hurewicz, Ch. 4. See note 8.
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